GENERALIZED ITÔ INTEGRAL AND HENSTOCK-YOUNG INTEGRAL

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Generalized Itô Integral and Henstock-Young Integral

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Contents

Summary

The Itô integral is an integral of adapted processes with respect to a Brownian motion. It is an integral of Stieltjes-type. Unfortunately, paths of a Brownian motion are of unbounded variation on a compact interval. Hence the classical measure and integration theory cannot be applied to the Itô integral. K. Itô defined his integral in 1944 by Cauchy sequences of integrals of simple processes. Thus his integral is of Bochner-type or of Riesz-type. This approach is less intuitive than that of the Riemann-Stieltjes approach.

Recently, a modified Riemann-Stieltjes approach has been successfully applied to the Itô integral and other stochastic integrals. It can be done since, in this modified approach, Riemann sums are induced by nonuniform meshes, whereas in the classical Riemann-Stieltjes approach, Riemann sums are induced by uniform meshes. The idea of nonuniform meshes was introduced by Henstock and Kurzweil independently in 1950's.

In this thesis, we shall again use the modified Riemann-Stieltjes approach to investigate integrals of nonadapted processes with respect to a Brownian motion. Furthermore, we also use this approach to investigate integrals with integrators of unbounded variation.

The main chapters in this thesis are Chapters three and four. In Chapter three, we define a new integral of processes with respect to a Brownian motion, without assuming that processes are adapted. Recall that in the Itô integral, processes are assumed to be adapted. Our new integral and the Itô integral have similar properties, and they are equivalent if an integrand is adapted.

In Chapter four, we use an idea of L.C. Young to define integrals with integrators of unbounded variation but of p-variation. The approach again is the modified Riemann-Stieltjes approach. The integrals can be used to handle stochastic integrals since paths of fractional Brownian motions are of p-variation.

Chapter two is an auxiliary chapter for Chapter three. Chapter one is a collection of basic concepts and known results needed in this thesis.

Chapter 1

Preliminaries

In this chapter, we collect some basic concepts and known results needed in this thesis.

The reader is referred to [6, 18, 19] for Sections 1.2-1.4 and Section 1.6, and to [2, 10, 11, 12, 15, 17, 20] for Sections 1.5 and 1.7.

1.1 Notations

Throughout this thesis, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of positive real numbers, $\mathbb{R}_0^+$ denotes $\mathbb{R}^+ \cup \{0\}$, and $\mathbb{N}$ denotes the set of natural numbers.

1.2 Probability space

Throughout this thesis, $(\Omega, \mathcal{F}, \mathbf{P})$ denotes a complete probability space. This means that $(\Omega, \mathcal{F})$ is a measurable space and $\mathbf{P}$ is a probability measure (i.e., $\mathbf{P}(\Omega) = 1$) on $(\Omega, \mathcal{F})$ such that each subset of a $\mathbf{P}$-null set in $\mathcal{F}$ is in $\mathcal{F}$.

The abbreviation "a.s." for "almost surely" means "$\mathbf{P}$-a.e.".

Throughout this thesis, $(\Omega, \mathcal{F}, \mathbf{P})$ is fixed.

We write $L^p(\Omega)$ for $L^p(\Omega, \mathcal{F}, \mathbf{P})$. For $f \in L^1(\Omega)$, $\mathbf{E}(f)$ denotes the expectation of f, i.e., $\mathbf{E}(f) = \displaystyle\int_\Omega f \, d\mathbf{P}$.

An $\mathcal{F}$-measurable function $f : \Omega \to \mathbb{R}$ is called a random variable.

If $f : \Omega \to \mathbb{R}$ is a random variable, then the σ-algebra $\sigma(f)$ generated by f is the smallest σ-algebra on Ω containing all subsets $f^{-1}(G)$, where G is an open set in $\mathbb{R}$.

Two subsets $A, B \in \mathcal{F}$ are called independent if $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$. A collection $\mathcal{A} = \{\mathcal{H}_i : i \in I\}$ of families $\mathcal{H}_i$ of measurable sets is independent if $\mathbf{P}(H_{i_1} \cap H_{i_2} \cap ... \cap H_{i_k}) = \mathbf{P}(H_{i_1})\mathbf{P}(H_{i_2}) \cdots \mathbf{P}(H_{i_k})$ for all choices of $H_{i_i} \in \mathcal{H}_{i_1}, H_{i_2} \in \mathcal{H}_{i_2}, ..., H_{i_k} \in \mathcal{H}_{i_k}$ with different indices $i_1, i_2, ..., i_k$. A collection of random variables $\{f_i : i \in I\}$ is independent if the collection of induced σ-algebra $\sigma(f_i)$ is independent.

A process f is a function $f : \Omega \times I \to \mathbb{R}$, where I is an interval in $\mathbb{R}_0^+$ and $f(\cdot, t)$ is $\mathcal{F}$-measurable for each $t \in I$. The process f is also denoted by $\{f_t : t \in I\}$ or simply $\{f_t\}$. A process f is said to be an $L^p(\Omega)$-process if $\mathbf{E}(|f_t|^p) < \infty$ for each t.

A filtration is a family $\{\mathcal{F}_t : t \in \mathbb{R}_0^+\}$ of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s < t$ in $\mathbb{R}_0^+$. If the following two conditions are also satisfied, then $\{\mathcal{F}_t : t \in \mathbb{R}_0^+\}$ is called a standard filtration:

(i) $\mathcal{F}_t = \displaystyle\bigcap_{s>t} \mathcal{F}_s$;

(ii) $\mathcal{F}_0$ contains all of the $\mathbf{P}$-null sets in $\mathcal{F}$.

We often write $\{\mathcal{F}_t\}$ instead of $\{\mathcal{F}_t : t \in \mathbb{R}_0^+\}$.

A process f is said to be adapted with respect to $\{\mathcal{F}_t : t \in \mathbb{R}_0^+\}$ if f_t is $\mathcal{F}_t$-measurable for each $t \in I$.

1.3 Brownian motion

A process $\{B : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}\}$ is called a Brownian motion if it has the following properties:

(i) (Normal increments) for $0 \leq s < t < \infty$, $B_t - B_s$ is a normally distributed random variable with mean zero and variance $t - s$;

(ii) (Independence of increments) for $0 \leq t_0 < t_1 < ... < t_n < \infty$,

$$\{B_{t_0}; B_{t_k} - B_{t_{k-1}}, k = 1, 2, ..., n\}$$

is a set of independent random variables.

If the following two conditions are also satisfied, then B is called a standard Brownian motion

(iii) (Continuity of paths) for fixed $\omega \in \Omega$, $B_t(\omega)$ is a continuous function of t ;

(iv) $B_0(\omega) = 0$ for all $\omega \in \Omega$.

It is known that for any fixed ω, $B_t(\omega)$ is a nowhere differentiable function of t, and of unbounded variation on any compact interval $[0, a]$.

Let $\sigma(B_s; s \leq t)$ be the smallest σ-algebra induced by $\{B_s; s \leq t\}$. This is the smallest σ-algebra containing the information about the structure of the Brownian motion on $[0, t]$. Throughout this thesis, we always denote $\sigma(B_s; s \leq t)$ by $\mathcal{F}_t$. In other words, the standard filtration used in this thesis is $\{\sigma(B_s : s \leq t); t \in \mathbb{R}_0^+\}$.

Hence B is adapted to $\{\mathcal{F}_t\}$ in this thesis. We remark that for $0 < s < t$, $B_t - B_s$ and $B_u - B_0 = B_u$ are independent for all $u \leq s$. Thus $B_t - B_s$ is independent of $\mathcal{F}_s = \sigma(B_u : u \leq s)$, i.e., $\sigma(B_t - B_s)$ and $\sigma(B_u : u \leq s)$ are independent.

1.4 Conditional expectation

Let $f \in L^1(\Omega)$. Then the *conditional expectation* of f given $\mathcal{F}_t$ is defined to be a process $\mathbf{E}[f|\mathcal{F}_t]$ such that

(i) $\mathbf{E}[f|\mathcal{F}_t]$ is $\mathcal{F}_t$-measurable, and

(ii) for any $A \in \mathcal{F}_t$

$$\int_A \mathbf{E}[f|\mathcal{F}_t] \, d\mathbf{P} = \int_A f \, d\mathbf{P}.$$

Theorem 1.4.1. *Let $\alpha \in \mathbb{R}$ and f, g be in $L^1(\Omega)$. Then, for any, $t, s \in \mathbb{R}_0^+$ with $s < t$,*

(i) $\mathbf{E}[f + g|\mathcal{F}_t] = \mathbf{E}[f|\mathcal{F}_t] + \mathbf{E}[g|\mathcal{F}_t]$ *;*

(ii) $\mathbf{E}[\alpha f|\mathcal{F}_t] = \alpha \mathbf{E}[f|\mathcal{F}_t]$ *;*

(iii) $\mathbf{E}[f|\mathcal{F}_t] \leq \mathbf{E}[g|\mathcal{F}_t]$, *if* $f \leq g$ *;*

(iv) $\mathbf{E}(f|\mathcal{F}_t) = f$, *if* f *is* $\mathcal{F}_t$*-measurable ;*

(v) $\mathbf{E}[\mathbf{E}[f|\mathcal{F}_t]|\mathcal{F}_s] = \mathbf{E}[f|\mathcal{F}_s]$ *;*

(vi) $\mathbf{E}[\mathbf{E}(f|\mathcal{F}_t)] = \mathbf{E}(f)$ *;*

(vii) $\mathbf{E}(f|\mathcal{F}_t) = \mathbf{E}(f)$, *if* f *is independent of* $\mathcal{F}_t$ *;*

(viii) $\mathbf{E}(f \cdot g|\mathcal{F}_t) = f \cdot \mathbf{E}(g|\mathcal{F}_t)$ *if* f *is* $\mathcal{F}_t$*-measurable.*

Theorem 1.4.2 (Jensen's Inequality)**.** *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function and $f, \varphi(f)$ be in $L^1(\Omega)$. Then*

$$\varphi(\mathbf{E}(f|\mathcal{G})) \leq \mathbf{E}(\varphi(f)|\mathcal{G}) \ \text{a.s.}$$

for any σ-field $\mathcal{G}$ on Ω contained in $\mathcal{F}$.

Lemma 1.4.3. *If $u < v$ and f is $\mathcal{F}_u$-measurable, then*

(i) $\mathbf{E}(B_v - B_u)^2 = v - u$;

(ii) $\mathbf{E}(B_v - B_u)^4 = 3(v - u)^2$;

(iii) $\mathbf{E}((B_v - B_u)|\mathcal{F}_u) = 0$;

(iv) $\mathbf{E}(B_v|\mathcal{F}_u) = B_u$;

(v) $\mathbf{E}((B_v - B_u)^2|\mathcal{F}_u) = v - u$;

(vi) $\mathbf{E}(f(B_v - B_u)^2) = \mathbf{E}(f(B_v^2 - B_u^2)) = \mathbf{E}(f)(v - u)$ *and* $\mathbf{E}(B_u - B_v)^2 = \mathbf{E}\ (B_v^2 - B_u^2)$;

(vii) $\mathbf{E}\big(f(B_v - B_u)(B_t - B_s)\big) = 0$ *and* $\mathbf{E}\big((B_v - B_u)(B_t - B_s)\big) = 0$, *where* $s < t \leq u < v$;

(viii) $\mathbf{E}(\sum_{i=1}^{n}(B_{v_i} - B_{u_i}))^2 = \sum_{i=1}^{n}\mathbf{E}(B_{v_i} - B_{u_i})^2$, *where* $\{[u_i, v_i]\}_{i=1}^{n}$ *are non-overlapping subintervals of* $[0, \infty)$.

Proof. We shall give proofs here for our own reference.

(i) It follows from property (i) of a Brownian motion ;

(ii) we consider the moment generating function : $M_X(q) = E(e^{qX})$. Recall that $B_v - B_u$ follows a normal distribution with mean zero and variance $v - u$, where $v > u$. Hence $M_X(q) = e^{(v-u)\frac{q^2}{2}}$, where $X = B_v - B_u$. Then the r^{th}-derivative of M_X at $q = 0$ is $\mathbf{E}(X^r) = M_X^{(r)}(0)$. Therefore $\mathbf{E}(B_v - B_u)^4 = 3(v - u)^2$;

(iii) if $u < v$, then $B_v - B_u$ is independent of $\mathcal{F}_u$, By Theorem 1.4.1 (vii),

$$\mathbf{E}((B_v - B_u)|\mathcal{F}_u) = \mathbf{E}(B_v - B_u) = 0;$$

(iv) if $u < v$, then, by Theorem 1.4.1 (i), (iv) and the above result (iii), we get

$$\begin{aligned}\mathbf{E}(B_v|\mathcal{F}_u) =& \mathbf{E}((B_v - B_u)|\mathcal{F}_u) + \mathbf{E}(B_u|\mathcal{F}_u)\\ =& 0 + B_u\\ =& B_u;\end{aligned}$$

(v) It is known that $B_v - B_u$ is independent of $\mathcal{F}_u$. Hence $(B_v - B_u)^2$ is independent of $\mathcal{F}_u$. By Theorem 1.4.1 (vii),

$$\mathbf{E}((B_v - B_u)^2|\mathcal{F}_u) = \mathbf{E}(B_v - B_u)^2 = v - u;$$

(vi)

$$\mathbf{E}(f(B_v - B_u)^2) = \mathbf{E}(f\mathbf{E}((B_v - B_u)^2|\mathcal{F}_u)) = \mathbf{E}(f)(v - u)$$

and

$$\begin{aligned}\mathbf{E}(f(B_v - B_u)^2) =& \mathbf{E}(f(B_v^2 - 2B_vB_u + B_u^2))\\ =& \mathbf{E}(f(B_v^2 + B_u^2)) - 2\mathbf{E}(fB_vB_u)\\ =& \mathbf{E}(f(B_v^2 + B_u^2)) - 2\mathbf{E}\big(\mathbf{E}(fB_vB_u)|\mathcal{F}_u\big)\\ =& \mathbf{E}(f(B_v^2 + B_u^2)) - 2\mathbf{E}\big(fB_u\mathbf{E}(B_v|\mathcal{F}_u)\big)\\ =& \mathbf{E}(f(B_v^2 + B_u^2)) - 2\mathbf{E}(fB_u^2)\\ =& \mathbf{E}(f(B_v^2 - B_u^2)).\end{aligned}$$

Note that if $g(\omega) = 1$ for all ω, then the inverse image of any interval under g is either ϕ or Ω. Hence g is $\mathcal{F}_u$-measurable for any u. Thus we get the second equality of (vi) ;

(vii)

$$\mathbf{E}(f(B_v - B_u)(B_t - B_s)) = \mathbf{E}(f(B_t - B_s)\mathbf{E}((B_v - B_u)|\mathcal{F}_u)) = 0 \text{ by (iii) ;}$$

Let $f(\omega) = 1$ for all ω. Then f is $\mathcal{F}_u$-measurable. Hence we get the second equality from the above ;

(viii) it follows form (vii).

□

1.5 The Bochner integral

The Bochner integral is an integral of functions with values in Banach space X. Throughout this thesis, $X = L^1(\Omega)$.

Let $f : \Omega \times [a, b] \to \mathbb{R}$ be a process. A process f is said to be simple if there exist $g^{(1)}, g^{(2)}, ..., g^{(n+1)} \in L^1(\Omega)$ and $a = t_1 < t_2 < ... < t_{n+1} = b$ such that $f = \sum_{i=1}^{n} g^{(i)} \mathcal{X}_{[t_i, t_{i+1})} + g^{(n+1)} \mathcal{X}_{\{b\}}$ where $\mathcal{X}_{[t_i, t_{i+1})}$ is the characteristic function of $[t_i, t_{i+1})$. The Bochner integral of a simple process f as above on $[a, b]$ is defined to be $\sum_{i=1}^{n} g^{(i)} |t_{i+1} - t_i|$. In this section, we only consider processes f which are almost everywhere pointwise limit of simple processes.

Definition 1.5.1. A process f is said to be Bochner integrable on $[a, b]$ if there exists a sequence of simple processes $\{f^{(n)}\}$ such that

$$\lim_{n \to \infty} (L) \int_a^b \mathbf{E}(|f_t^{(n)} - f_t|) \, dt = 0,$$

where the integral $(L) \int_a^b \mathbf{E}(|f_t^{(n)} - f_t|) \, dt$ used above is the Lebesgue integral. If f is Bochner integrable on $[a, b]$, then the Bochner integral of f on $[a, b]$ is defined to be $\lim_{n \to \infty} \int_a^b f^{(n)}$, where $\int_a^b f^{(n)}$ is the Bochner integral of the simple process $f^{(n)}$ defined above. It is known that the Bochner integral is independent of the defining sequence $\{f^{(n)}\}$.

1.6 The Itô integral

Let f be a simple $L^2(\Omega)$-process on $[a, b]$, i.e., $f : \Omega \times [a, b] \to \mathbb{R}$ and

$$f = \sum_{i=1}^{n} g^{(i)} \mathcal{X}_{[t_i, t_{i+1})} + g^{(n+1)} \mathcal{X}_{\{b\}}.$$

where $a = t_1 < t_2 < ... < t_{n+1} = b$ and $g^{(i)} \in L^2(\Omega)$ for each i. In this section, we always assume that f is adapted, i.e., for each i, $g^{(i)}$ is $\mathcal{F}_{t_i}$-measurable. The Itô integral of f is defined to be

$$\sum_{i=1}^{n} g^{(i)} (B_{t_{i+1}} - B_{t_i})$$

which is denoted by $(It\hat{o}) \int_a^b f_t \, dB_t$, $(It\hat{o}) \int_a^b f \, dB$ or $(It\hat{o}) \int_a^b f$. Note that

$$\mathbf{E}\big((It\hat{o}) \int_a^b f\big)^2 = \sum_{i=1}^{n} \mathbf{E}(g^{(i)})^2 (t_{i+1} - t_i) = (L) \int_a^b \mathbf{E}(f_t^2) \, dt,$$

by Lemma 1.4.3 (vi). Now, let $f : \Omega \times [a, b] \to \mathbb{R}$ be an adapted $L^2(\Omega)$-process such that $(L) \int_a^b \mathbf{E}(|f_t|^2) \, dt$ exists. Then there exists a sequence $\{g^{(n)}\}$ of simple adapted $L^2(\Omega)$-processes such that $\lim_{n\to\infty} (L) \int_a^b \mathbf{E}(|g_t^{(n)} - f_t|^2) \, dt = 0$.

Therefore

$$\lim_{n\to\infty,\, m\to\infty} (L) \int_a^b \mathbf{E}((g_t^{(n)} - g_t^{(m)})^2) \, dt = 0.$$

On the other hand

$$\mathbf{E}\big((It\hat{o}) \int_a^b (g^{(n)} - g^{(m)})\big)^2 = (L) \int_a^b \mathbf{E}\big((g_t^{(n)} - g_t^{(m)})^2\big) \, dt$$

Hence $\{(It\hat{o}) \int_a^b g^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\Omega)$. Therefore its limit exists in $L^2(\Omega)$. The Itô integral $(It\hat{o}) \int_a^b f$ of f on $[a, b]$ is defined to be this limit. In other words

$$\lim_{n\to\infty} \mathbf{E}\big((It\hat{o}) \int_a^b g^{(n)} - (It\hat{o}) \int_a^b f\big)^2 = 0.$$

It is well-known that the following isometry property holds:

$$\mathbf{E}\big((It\hat{o})\int_a^b f\big)^2 = (L)\int_a^b \mathbf{E}(f_t^2)\ dt.$$

1.7 The Henstock and McShane integrals

Let $P = \{[u_i, v_i]\}_{i=1}^n$ be a finite collection of non-overlapping subintervals of $[a, b]$. Then P is said to be a partial partition of $[a, b]$. In addition, if $\bigcup_{i=1}^n [u_i, v_i] = [a, b]$, then P is said to be a partition of $[a, b]$.

Let δ be a positive function on $[a, b]$, $[u, v] \subseteq [a, b]$ and $t \in [a, b]$, then an interval-point pair $([u, v], t)$ is said to be McShane δ-fine if $[u, v] \subseteq (t - \delta(t), t + \delta(t))$. In addition, if $t \in [u, v]$, then $([u, v], t)$ is said to be Henstock δ-fine or simply δ-fine. An interval-point pair $([u, v], t)$ is said to be belated δ-fine if $[u, v] \subseteq [t, t + \delta(t))$.

Let $D = \{([u_i, v_i], t_i)\}_{i=1}^n$ be a finite collection of interval-point pairs.

Then D is said to be a δ-fine partial McShane division of $[a, b]$ if $\{[u_i, v_i]\}_{i=1}^n$ is a partial partition of $[a, b]$ and for each i, $([u_i, v_i], t_i)$ is McShane δ-fine. In addition, if $\{[u_i, v_i]\}_{i=1}^n$ is a partition of $[a, b]$, then D is said to be a δ-fine McShane division of $[a, b]$.

Similarly, we can define δ-fine partial (Henstock) divisions and δ-fine (Henstock) divisions of $[a, b]$.

D is said to be δ-fine partial belated division of $[a, b]$ if $\{[u_i, v_i]\}_{i=1}^n$ is a partial partition of $[a, b]$, and for each i, $([u_i, v_i], t_i)$ is belated δ-fine. We remark that a δ-fine (full) belated division of $[a, b]$ may not exist. For example, if $\delta(t) = \frac{b-t}{2}$ if $t \neq b$, then a δ-fine (full) belated division of $[a, b]$ does not exist.

Definition 1.7.1. Let $f : [a, b] \to \mathbb{R}$. Then f is said to be Henstock integrable to A on $[a, b]$ if for each $\epsilon > 0$, there exists a positive function δ on $[a, b]$ such that whenever $D = \{([u_i, v_i], t_i)\}_{i=1}^n$ is a δ-fine (Henstock) division of $[a, b]$, we have

$$|\sum_{i=1}^{n} f(t_i)(v_i - u_i) - A| \leq \epsilon.$$

Definition 1.7.2. Let $f : [a, b] \to \mathbb{R}$. Then f is said to be McShane integrable to A on $[a, b]$ if for each $\epsilon > 0$, there exists a positive function δ on $[a, b]$ such that whenever $D = \{([u_i, v_i], t_i)\}_{i=1}^n$ is a δ-fine McShane division of $[a, b]$, we have

$$|\sum_{i=1}^{n} f(t_i)(v_i - u_i) - A| \leq \epsilon.$$

Definition 1.7.3. Let $f : [a, b] \to \mathbb{R}$. Then f is said to be belated integrable to A on $[a, b]$ if for each $\epsilon > 0$, there exist a positive function δ on $[a, b]$ and a positive number η such that whenever $D = \{([u_i, v_i], t_i)\}_{i=1}^n$ is a δ-fine partial belated division of $[a, b]$ with

$$|\ b - a - \sum_{i=1}^{n}(v_i - u_i)| \leq \eta,$$

we have

$$|\sum_{i=1}^{n} f(t_i)(v_i - u_i) - A| \leq \epsilon.$$

We remark that for any positive function δ and a positive number η, there exists a δ-fine partial belated division $D = \{([u_i, v_i], t_i)\}_{i=1}^n$ with $|\ b-a-\sum_{i=1}^{n}(v_i-u_i)| \leq \eta$, in view of Vitali's covering theorem.

It is easy to see that if f is McShane integrable on $[a, b]$, then f is Henstock integrable on $[a, b]$. It is known that there exists a Henstock integrable function which is not McShane integrable.

Theorem 1.7.4. *Let $f : [a, b] \to \mathbb{R}$. Then f is McShane integrable on $[a, b]$ if and only if f is Lebesgue integrable on $[a, b]$. Furthermore, their integrals are equal.*

Theorem 1.7.5. *[14] Let $f : [a, b] \to \mathbb{R}$. Then f is Lebesgue integrable on $[a, b]$ if and only if f is belated integrable on $[a, b]$. Furthermore, their integrals are equal.*

Chapter 2

The Generalized Henstock Integral

In this chapter, we shall discuss integrals of processes, which are approximated by Riemann sums $(D)\sum_i \mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})(t_{i+1} - t_i)$ or $(D)\sum_i \big(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})\big)^2(t_{i+1} - t_i)$ under L^1-norm. These integrals are closely related to integrals of processes, which are approximated by Riemann sums $(D)\sum_i \mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})(B_{t_{i+1}} - B_{t_i})$ under L^2-norm. They are closely related, since $\mathbf{E}(B_{t_{i+1}} - B_{t_i})^2 = t_{i+1} - t_i$.

2.1 Definition of the GH-integral

In this section, we shall define the generalized Henstock integral (henceforth abbreviated to GH-integral) GH-integral of a process, which is approximated by Riemann sum $(D)\sum_i \mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})(t_{i+1} - t_i)$ under L^1-norm. Observe that if f is deterministic, then $\mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i}) = f_{\xi_i}$ Hence, in this case, the GH-integral is deduced to the usual Henstock integral.

Definition 2.1.1 (The generalized Henstock integral)**.** Let $f : \Omega \times [a, b] \to \mathbb{R}$ be an $L^1(\Omega)$-process and $F \in L^1(\Omega)$. Then f is said to be *generalized Henstock integrable* (or GH-integrable) to F on $[a, b]$ if for every $\epsilon > 0$, there exists a positive function δ defined on $[a, b]$ such that for every δ-fine division $D = \{([t_i, t_{i+1}], \xi_i)\}_{i=1}^n$ of $[a, b]$, we have

$$\mathbf{E}(|\tilde{S}(f,\delta,D) - F|) \leq \epsilon,$$

where

$$\tilde{S}(f,\delta,D) = \sum_{i=1}^{n} \mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})(t_{i+1} - t_i).$$

We denote F by $(GH)\int_a^b f_t\ dt$, $(GH)\int_a^b f\ dt$ or $(GH)\int_a^b f$.

Moreover, let f be an $L^2(\Omega)$-process. Then f is said to be square GH-integrable to F on $[a,b]$ if $\tilde{S}$ is replaced by S' in the above definition, where

$$S'(f,\delta,D) = \sum_{i=1}^{n} [\mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})]^2(t_{i+1} - t_i).$$

We denote F by $(GH)\int_a^b f_t^{[2]}\ dt$, $(GH)\int_a^b f^{[2]}\ dt$ or $(GH)\int_a^b f^{[2]}$.

The following three theorems and their proofs are completely similar to the corresponding results and proofs in the theory of classical Henstock integration, see [2, 10, 11, 12, 20].

Theorem 2.1.2. *The integral F in Definition 2.1.1 is unique up to a set of* $\mathbf{P}$-*measure zero.*

Proof. We shall only prove the case for the generalized Henstock integral. Let $\epsilon > 0$. Assume that F_1 and F_2 satisfy the conditions in Definition 2.1.1, i.e., there exist δ_i such that for every δ_i-fine divisions D_i, where $i = 1, 2$, of $[a,b]$, we have

$$\mathbf{E}(|\tilde{S}(f,\delta_1,D_1) - F_1|) \leq \frac{\epsilon}{2},$$

and

$$\mathbf{E}(|\tilde{S}(f,\delta_2,D_2) - F_2|) \leq \frac{\epsilon}{2}.$$

Pick $\delta = \min\{\delta_1, \delta_2\}$, then a δ-fine division of $[a, b]$ is also a δ_1-fine division and a δ_2-fine division of $[a, b]$. Hence

$$\begin{aligned}\mathbf{E}(|F_2 - F_1|) =& \mathbf{E}(|(\tilde{S}(f, \delta, D) - F_1) - (\tilde{S}(f, \delta, D) - F_2)|) \\ \leq& \mathbf{E}(|\tilde{S}(f, \delta, D) - F_1|) + \mathbf{E}(|\tilde{S}(f, \delta, D) - F_2|) \\ =& \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Hence $\mathbf{E}(|F_1 - F_2|) = 0$. We can conclude that $F_1 = F_2$ almost surely, i.e., except only on a set of $\mathbf{P}$-measure zero. □

Theorem 2.1.3. *Let $f : \Omega \times [a, b] \to \mathbb{R}$ be an $L^1(\Omega)$-process. Then f is GH-integrable on $[a, b]$ if and only if there exist $F \in L^1(\Omega)$ and a decreasing sequence $\{\delta_n(\xi)\}_{n\in\mathbb{N}}$ of positive functions defined on $[a, b]$ such that we have*

$$\lim_{n\to\infty} \mathbf{E}(|\tilde{S}(f, \delta_n, D_n) - F|) = 0,$$

for any δ_n-fine divisions D_n, n=1,2,....

Proof. Let $\epsilon > 0$ be given. Assume that f is GH-integrable on $[a, b]$, then there exists a positive function δ on $[a, b]$ such that for every δ-fine division D of $[a, b]$, we have

$$\mathbf{E}(|\tilde{S}(f, \delta, D) - F|) \leq \epsilon.$$

Then there is a decreasing sequence $\{\delta_n(\xi)\}_{n\in\mathbb{N}}$ of positive functions such that for any δ_n-fine division D_n on $[a, b]$, we have

$$\mathbf{E}(|\tilde{S}(f, \delta_n, D_n) - F|) \leq \frac{1}{n},$$

thus, we can conclude that

$$\lim_{n\to\infty} \mathbf{E}(|\tilde{S}(f, \delta_n, D_n) - F|) = 0.$$

Conversely, assume that there exist $F \in L^1(\Omega)$ and a decreasing sequence $\{\delta_n(\xi)\}_{n\in\mathbb{N}}$ of positive functions on $[a, b]$ such that for any δ_n-fine divisions D_n, $n = 1, 2, ...$,

$$\lim_{n\to\infty} \mathbf{E}(|\tilde{S}(f, \delta_n, D_n) - F|) = 0.$$

Suppose that f is not GH-integrable on $[a,b]$. Then there exists $\epsilon > 0$ such that for every positive function δ on $[a,b]$, there exists a δ-fine division D of $[a,b]$ with

$$\mathbf{E}(|\tilde{S}(f,\delta,D) - F|) \geq \epsilon.$$

Hence for each δ_n, there exists a δ_n-fine division D_n of $[a,b]$ such that

$$\mathbf{E}(|\tilde{S}(f,\delta_n,D_n) - F|) \geq \epsilon.$$

It contradicts to the fact that $\lim_{n\to\infty} \mathbf{E}(|\tilde{S}(f,\delta_n,D_n) - F|) = 0$. Then we can conclude that f is GH-integrable on $[a,b]$. □

Similarly, we have

Theorem 2.1.4. *Let $f : \Omega \times [a,b] \to \mathbb{R}$ be an $L^2(\Omega)$-process. Then f is square GH-integrable on $[a,b]$ if and only if there exist $F \in L^2(\Omega)$ and a decreasing sequence $\{\delta_n(\xi)\}_{n\in\mathbb{N}}$ of positive functions defined on $[a,b]$ such that we have*

$$\lim_{n\to\infty} \mathbf{E}(|\tilde{S}(f,\delta_n,D_n) - F|) = 0,$$

for any δ_n-fine divisions D_n, n=1,2,....

2.2 Basic properties

In this section, we shall prove some basic properties of the generalized Henstock integral and establish the Cauchy Criterion for the generalized Henstock integral. The ideas of the proofs are similar to that of classical Henstock integrals, see [2, 10, 11, 12, 20]. There are two proofs we would like to highlight, which are the proofs of Theorems 2.2.6 and 2.2.8, where δ' functions are different from that in the classical case and Jensen's inequality for conditional expectations are used.

Theorem 2.2.1. *Let $\alpha \in \mathbb{R}$. If $f, g : \Omega \times [a,b] \to \mathbb{R}$ are GH-integrable processes on $[a,b]$, then* *(i) $f+g$ is GH-integrable on $[a,b]$, and*

$$(GH)\int_a^b (f_t+g_t)\ dt = (GH)\int_a^b f_t\ dt + (GH)\int_a^b g_t\ dt,$$

(ii) αf is GH-integrable on $[a,b]$, and

$$(GH)\int_a^b (\alpha f_t)\ dt = \alpha\cdot(GH)\int_a^b f_t\ dt.$$

Proof. Let $\epsilon>0$ and $\alpha\in\mathbb{R}$. Assume that f and g are GH-integrable processes on $[a,b]$ such that

$$(GH)\int_a^b f_t\ dt = F,$$

and

$$(GH)\int_a^b g_t\ dt = G.$$

Then there exist positive functions δ_1 and δ_2 on $[a,b]$ such that

$$\mathbf{E}(|\tilde{S}(f,\delta_1,D_1)-F|)\le\frac{\epsilon}{2},$$

and

$$\mathbf{E}(|\tilde{S}(g,\delta_2,D_2)-G|)\le\frac{\epsilon}{2}$$

for every δ_1,δ_2-fine divisions D_1,D_2 of $[a,b]$, respectively.

Pick $\delta(\xi)=\min\{\delta_1(\xi),\delta_2(\xi)\}$. Then for every δ-fine division $D=\{([t_i,t_{i+1}],\xi_i)\}_{i=1}^n$ of $[a,b]$, we have, $|\mathbf{E}(\tilde{S}(f,\delta,D)-F)|\le\frac{\epsilon}{2}$ and $|\mathbf{E}(\tilde{S}(g,\delta,D)-G)|\le\frac{\epsilon}{2}$.

Since

$$\begin{aligned}\tilde{S}(f+g,\delta,D) =&(D)\sum_{i=1}^n \mathbf{E}(f_{\xi_i}+g_{\xi_i}|\mathcal{F}_{t_i})(t_{i+1}-t_i)\\ =&(D)\sum_{i=1}^n \mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})(t_{i+1}-t_i)\\ &+(D)\sum_{i=1}^n \mathbf{E}(g_{\xi_i}|\mathcal{F}_{t_i})(t_{i+1}-t_i)\\ =&\tilde{S}(f,\delta,D)+\tilde{S}(g,\delta,D).\end{aligned}$$

Then,

$$\begin{aligned}\mathbf{E}(|\tilde{S}(f+g,\delta,D)-(F+G)|) =&\mathbf{E}(|(\tilde{S}(f,\delta,D)-F)+(\tilde{S}(g,\delta,D)-G)|)\\ \leq&\mathbf{E}(|\tilde{S}(f,\delta,D)-F|)+\mathbf{E}(|\tilde{S}(g,\delta,D)-G|)\\ \leq&\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

Hence, $f+g$ is GH-integrable on $[a,b]$ and

$$\begin{aligned}(GH)\int_a^b (f_t+g_t)\ dt =&F+G\\ =&(GH)\int_a^b f_t\ dt+(GH)\int_a^b g_t\ dt.\end{aligned}$$

Next, consider the case for αf. It is obvious that

$$(GH)\int_a^b 0\cdot f_t\ dt=0=0\cdot(GH)\int_a^b f_t\ dt.$$

Now, assume that $\alpha\neq 0$. There exists a positive function δ on $[a,b]$ such that for every δ-fine division D of $[a,b]$

$$\mathbf{E}(|\tilde{S}(f,\delta,D)-F|)\leq\frac{\epsilon}{\alpha}.$$

Since

$$\begin{aligned}\tilde{S}(\alpha f,\delta,D) =&(D)\sum_{i=1}^{n}\mathbf{E}(\alpha f_{\xi_i}|\mathcal{F}_{t_i})(t_{i+1}-t_i)\\ =&\alpha(D)\sum_{i=1}^{n}\mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})(t_{i+1}-t_i)\\ =&\alpha\tilde{S}(f,\delta,D).\end{aligned}$$

Then,

$$\begin{aligned}\mathbf{E}(|\tilde{S}(\alpha f,\delta,D)-\alpha F|) =&\mathbf{E}(|\alpha\tilde{S}(f,\delta,D)-\alpha F|)\\ =&\alpha\mathbf{E}(|\tilde{S}(f,\delta,D)-F|)\\ \leq&\alpha(\frac{\epsilon}{\alpha})=\epsilon.\end{aligned}$$

Hence, αf is GH-integrable on $[a, b]$ and

$$(GH)\int_a^b \alpha f_t\ dt = \alpha F = \alpha \cdot (GH)\int_a^b t\ dt.$$

□

Definition 2.2.2. Let $f : \Omega \times [a, b] \to \mathbb{R}$ be an $L^1(\Omega)$-process, $A \subseteq [a, b]$ and $\mathcal{X}_A$ be the characteristic function of A. Then f is said to be GH-integrable on A if $f \cdot \mathcal{X}_A$ is GH-integrable on $[a, b]$. We denote $(GH)\int_a^b f \cdot \mathcal{X}_A\ dt$ by

$$(GH)\int_A f\ dt.$$

We remark that if $A = \{c\}$, where $c \in [a, b]$, then $f \cdot \mathcal{X}_A$ is GH-integrable on $[a, b]$ and $(GH)\int_a^b f \cdot \mathcal{X}_A\ dA = 0$.

Theorem 2.2.3. *Let $c \in (a, b)$. If $f : \Omega \times [a, b] \to \mathbb{R}$ is a GH-integrable process on [a,c] and [c,b], then f is GH-integrable on $[a, b]$ and*

$$(GH)\int_a^b f\ dt = (GH)\int_a^c f\ dt + (GH)\int_c^b f\ dt.$$

Proof. By Definition 2.2.2, we have

$$(GH)\int_a^c f\ dt = (GH)\int_a^b f \cdot \mathcal{X}_{[a,c]}\ dt,$$

and

$$(GH)\int_c^b f\ dt = (GH)\int_a^b f \cdot \mathcal{X}_{[c,b]}\ dt.$$

Then, from Theorem 2.2.1, we can see that

$$\begin{aligned}(GH)\int_a^b f\ dt =&(GH)\int_a^b \left(f \cdot \mathcal{X}_{[a,c]} + f \cdot \mathcal{X}_{(c,b]}\right)\ dt\\ =&(GH)\int_a^b f \cdot \mathcal{X}_{[a,c]}\ dt + (GH)\int_a^b f \cdot \mathcal{X}_{(c,b]}\ dt\\ =&(GH)\int_a^b f \cdot \mathcal{X}_{[a,c]}\ dt + (GH)\int_a^b f \cdot \mathcal{X}_{[c,b]}\ dt\\ =&(GH)\int_a^c f\ dt + (GH)\int_c^b f\ dt.\end{aligned}$$

□

Remark 2.2.4. *Theorems 2.2.1 and 2.2.3 can also be similarly proved for square GH-integrable processes except those equalities, for example*

$$(GH)\int_a^b (f_t+g_t)^{[2]}\ dt \neq (GH)\int_a^b f_t^{[2]}\ dt + (GH)\int_a^b g_t^{[2]}\ dt,$$

since

$$\big(\mathbf{E}(f_t+g_t|\mathcal{F}_u)\big)^2 \neq \big(\mathbf{E}(f_t|\mathcal{F}_u)\big)^2 + \big(\mathbf{E}(g_t|\mathcal{F}_u)\big)^2.$$

Theorem 2.2.5 (Cauchy Criterion for generalized Henstock integrals)**.** *Let $f : \Omega\times[a,b]\to\mathbb{R}$ be an $L^1(\Omega)$-process. Then f is GH-integrable on $[a,b]$ if and only if for every $\epsilon>0$, there exists a positive function δ on $[a,b]$ such that for any two δ-fine divisions of $[a,b]$, $D=\{([t_i,t_{i+1}],\xi_i)\}$ and $D'=\{([t'_j,t'_{j+1}],\xi'_j)\}$, we have*

$$\mathbf{E}(|\tilde{S}(f,\delta,D)-\tilde{S}(f,\delta,D')|)\leq\epsilon.$$

Proof. Let $f:\Omega\times[a,b]\to\mathbb{R}$ be an $L^1(\Omega)$-process. Assume that f is GH-integrable on $[a,b]$.

Let $\epsilon>0$ be given. Then there exists a positive function δ on $[a,b]$ such that for any δ-fine divisions D and D' of $[a,b]$, we have

$$\mathbf{E}(|\tilde{S}(f,\delta,D)-F|)\leq\frac{\epsilon}{2},$$

and

$$\mathbf{E}(|\tilde{S}(f,\delta,D')-F|)\leq\frac{\epsilon}{2}.$$

We can see that

$$\begin{aligned}\mathbf{E}(|\tilde{S}(f,\delta,D)-\tilde{S}(f,\delta,D')|) =& \mathbf{E}(|(\tilde{S}(f,\delta,D)-F)-(\tilde{S}(f,\delta,D')-F)|)\\ \leq& \mathbf{E}(|\tilde{S}(f,\delta,D)-F|)+\mathbf{E}(|\tilde{S}(f,\delta,D')-F|)\\ \leq& \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

Conversely, assume that for every $\epsilon>0$, there exists a positive function δ on $[a,b]$ such that for any two δ-fine divisions of $[a,b]$, $D=\{[t_i,t_{i+1}],\xi_i\}$ and

$D' = \{[t'_j, t'_{j+1}], \xi'_j\}$, we have

$$\mathbf{E}(|\tilde{S}(f,\delta,D) - \tilde{S}(f,\delta,D')|) \leq \frac{\epsilon}{2}.$$

Let $\epsilon_n = \frac{2}{n}$, for $n \in \mathbb{N}$ and δ_n be the corresponding positive function on $[a,b]$. We may assume that if $m \geq n$, $\delta_m(\xi) \leq \delta_n(\xi)$ for each $\xi \in [a,b]$. Then a δ_m-fine division of $[a,b]$ is also a δ_n-fine division of $[a,b]$. Hence

$$\mathbf{E}(|\tilde{S}(f,\delta_{m_1},D_{m_1}) - \tilde{S}(f,\delta_{m_2},D_{m_2})|) \leq \frac{1}{n}$$

whenever $m_1, m_2 \geq n$.

We conclude that $\{\tilde{S}(f,\delta_n,D_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\Omega)$-space. Therefore, there exists $F \in L^1(\Omega)$ such that $\lim\limits_{n\to\infty} \mathbf{E}(|\tilde{S}(f,\delta_n,D_n) - F|) = 0$.

Let $\epsilon > 0$ be given. Then there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$,

$$\mathbf{E}(|\tilde{S}(f,\delta_n,D_n) - F|) \leq \frac{\epsilon}{2}.$$

Let $\frac{1}{N_2} \leq \frac{\epsilon}{2}$ and $N = max\{N_1, N_2\}$. Let $\delta(\xi) = \delta_N(\xi)$ for each $\xi \in [a,b]$. Then

$$\begin{aligned}\mathbf{E}(|\tilde{S}(f,\delta,D) - F|) =& \mathbf{E}(|(\tilde{S}(f,\delta,D) - \tilde{S}(f,\delta_N,D_N)) + (\tilde{S}(f,\delta_N,D_N) - F)|) \\ \leq& \mathbf{E}(|\tilde{S}(f,\delta,D) - \tilde{S}(f,\delta_N,D_N)|) + \mathbf{E}(|\tilde{S}(f,\delta_N,D_N) - F|) \\ \leq& \frac{1}{N} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Thus f is GH-integrable on $[a,b]$. Moreover, we also know that $(GH)\int_a^b f\ dt = F$.
□

Theorem 2.2.6. *If f is GH-integrable on $[a,b]$, then f is also GH-integrable on every subinterval $[c,d]$ of $[a,b]$.*

Proof. Although the idea of the proof comes from the classical theory of Henstock integration. However, the δ' function in the following is different from the classical one because in the last part of the proof, we have to ensure that $|t_{g+1} - c|$ and

$|t'_{h+1} - c|$ are less that ϵ. We shall only prove the case that f is GH-integrable on $[a, c]$. Let f be GH-integrable on $[a, b]$. Let $\epsilon > 0$ be given. By Cauchy Criterion for generalized Henstock integrals, there exists a positive function δ on $[a, b]$ such that for any two δ-fine divisions D, D' of $[a, b]$, we have

$$\mathbf{E}(|\tilde{S}(f, \delta, D) - \tilde{S}(f, \delta, D')|) \leq \epsilon.$$

Choose

$$\delta'(\xi) = \begin{cases} \min\{\delta(\xi), c - \xi, \epsilon\} & , \text{ if } \xi \in [a, c); \\ \min\{\delta(\xi), \epsilon\} & , \text{ if } \xi = c; \\ \min\{\delta(\xi), \xi - c, \epsilon\} & , \text{ if } \xi \in (c, b]. \end{cases}$$

It is clear that $\delta'(\xi) \leq \delta(\xi)$. Hence, every δ'-fine division D is also δ-fine.

Let $D_1 = \{([t_i, t_{i+1}], \xi_i)\}_{i=1}^m$ and $D_2 = \{([t'_j, t'_{j+1}], \xi'_j)\}_{j=1}^n$ be δ'-fine divisions of $[a, b]$.

It is clear that c always is a tag, we may assume that $\xi_g = c = \xi'_h$. Let D'_1 be a partial division of D_1 by deleting $(c, t_{g+1}]$ and $\{[t_i, t_{i+1}]\}_{i=g+1}^m$ from D_1, we do the same thing for D_2 and obtain D'_2. Thus D'_1 and D'_2 form δ-fine divisions of $[a, c]$

Let D_3 be a δ-fine division of $[c, b]$. Hence $D'_1 \cup D_3$ and $D'_2 \cup D_3$ form δ-fine full divisions of $[a, b]$. Then,

$$\begin{aligned}
&\mathbf{E}(|\tilde{S}(f\mathcal{X}_{[a,c]}, \delta', D_1) - \tilde{S}(f\mathcal{X}_{[a,c]}, \delta', D_2)|) \\
=&\mathbf{E}(|\tilde{S}(f, \delta', D'_1) + \mathbf{E}(f_c|\mathcal{F}_{t_g})(t_{g+1} - c) - \tilde{S}(f, \delta', D'_2) - \mathbf{E}(f_c|\mathcal{F}_{t'_h})(t'_{h+1} - c)|) \\
\leq&\mathbf{E}(|\mathbf{E}(f_c|\mathcal{F}_{t_{g+1}})(t_{g+1} - c)| + |\mathbf{E}(f_c|\mathcal{F}_{t'_h})(t'_{h+1} - c)| \\
&+ \mathbf{E}(|\tilde{S}(f, \delta, D'_1 \cup D_3) - \tilde{S}(f, \delta, D'_2 \cup D_3)|) \\
\leq&2\epsilon\mathbf{E}(|f_c|) + \epsilon, \text{(by Jensen's inequality for conditional expectations (Theorem 1.4.2))}
\end{aligned}$$

Hence f is GH-integral on $[a, c]$.

Similarly, we can prove the case for $[d, b]$. Since $f\mathcal{X}_{[c,d]} = f - f\mathcal{X}_{[a,c)} - f\mathcal{X}_{(d,b]}$ and $\int_a^b f \cdot \mathcal{X}_{\{c\}}\, dt = \int_a^b f \cdot \mathcal{X}_{\{d\}}\, dt = 0$. Hence f is GH-integral on $[c, d]$. $\square$

Theorem 2.2.7 (Cauchy Criterion for square generalized Henstock integrals)**.** *Let $f : \Omega \times [a, b] \to \mathbb{R}$ be an $L^2(\Omega)$-process. Then f is square GH-integrable on $[a, b]$ if and only if for every $\epsilon > 0$, there exists a positive function δ on $[a, b]$ such that for any two δ-fine divisions of $[a, b]$, $D = \{[t_i, t_{i+1}], \xi_i\}$ and $D' = \{[t'_j, t'_{j+1}], \xi'_j\}$, we have*

$$\mathbf{E}(|S'(f, \delta, D) - S'(f, \delta, D')|) \leq \epsilon,$$

where

$$S'(f, \delta, D) = (D) \sum_i [\mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})]^2 (t_{i+1} - t_i).$$

Proof. The proof is similar to that of Theorem 2.2.5. □

Theorem 2.2.8. *If f is square GH-integrable on $[a, b]$, then f is also square GH-integrable on every subinterval $[c, d]$ of $[a, b]$.*

Proof. The proof is similar to that of Theorem 2.2.6. As in Theorem 2.2.6, we shall use Jensen's inequality for conditional expectations in the last part of the proof. We shall only prove the case that f is square GH-integrable on $[a, c]$. Let f be square GH-integrable on $[a, b]$. Let $\epsilon > 0$ be given. By Cauchy Criterion for generalized Henstock integrals, there exists a positive function δ on $[a, b]$ such that for any two δ-fine divisions D, D' of $[a, b]$, we have

$$\mathbf{E}(|S'(f, \delta, D) - S'(f, \delta, D')|) \leq \epsilon.$$

Choose

$$\delta'(\xi) = \begin{cases} \min\{\delta(\xi), c - \xi, \epsilon\} & , \text{ if } \xi \in [a, c); \\ \min\{\delta(\xi), \epsilon\} & , \text{ if } \xi = c; \\ \min\{\delta(\xi), \xi - c, \epsilon\} & ; \text{ if } \xi \in (c, b]. \end{cases}$$

It is clear that $\delta'(\xi) \leq \delta(\xi)$. Hence every δ'-fine division D is also δ-fine.

Let $D_1 = \{([t_i, t_{i+1}], \xi_i)\}_{i=1}^m$ and $D_2 = \{([t'_j, t'_{j+1}], \xi'_j)\}_{j=1}^n$ be δ'-fine divisions of $[a, b]$.

It is clear that c always is a tag, we may assume that $\xi_g = c = \xi'_h$. Let D'_1 be a partial division of D_1 by deleting $(c, t_{g+1}]$ and $\{[t_i, t_{i+1}]\}_{i=g+1}^m$ from D_1, we do the same thing for D_2 and obtain D'_2. Thus D'_1 and D'_2 form δ-fine divisions of $[a, c]$

Let D_3 is a δ-fine division of $[c, b]$. Hence $D'_1 \cup D_3$ and $D'_2 \cup D_3$ form δ-fine divisions of $[a, b]$. Then,

$$\begin{aligned}
&\mathbf{E}(|S'(f\mathcal{X}_{[a,c]}, \delta', D_1) - S'(f\mathcal{X}_{[a,c]}, \delta', D_2)|) \\
=&\mathbf{E}(|S'(f, \delta', D'_1) + [\mathbf{E}(f_c|\mathcal{F}_{t_g})]^2(t_{g+1} - c) - S'(f, \delta', D'_1) - [\mathbf{E}(f_c|\mathcal{F}_{t'_h})]^2(t'_{h+1} - c)|) \\
\leq&\mathbf{E}([\mathbf{E}(f_c|\mathcal{F}_{t_{g+1}})]^2(t_{g+1} - c)| + |[\mathbf{E}(f_c|\mathcal{F}_{t'_h})]^2(t'_{h+1} - c) \\
&+ \mathbf{E}(|S'(f, \delta, D'_1 \cup D_3) - S'(f, \delta, D'_2 \cup D_3)|) \\
\leq&2\epsilon\mathbf{E}(f_c^2) + \epsilon. \text{ (by Jensen's inequality for conditional expectations (Theorem 1.4.2))}
\end{aligned}$$

Hence f is square GH-integral on $[a, c]$.

Similarly, we can prove the case for $[d, b]$. Since $f\mathcal{X}_{[c,d]} = f - f\mathcal{X}_{[a,c)} - f\mathcal{X}_{(d,b]}$. Hence f is square GH-integral on $[c, d]$. □

Lemma 2.2.9 (Henstock's lemma)**.** *If f is GH-integrable on $[a, b]$ and $F(u, v) = (GH)\int_u^v f_t\, dt$ for any $[u, v] \subseteq [a, b]$, then for every $\epsilon > 0$, there exists a positive function δ on $[a, b]$ such that for every δ-fine partial division $D' = \{([u_i, v_i], \xi_i)\}$ of $[a, b]$, we have*

$$\mathbf{E}(|(D')\sum_{i=1}^{n} \mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(v_i - u_i) - F(u_i, v_i)|) \leq \epsilon.$$

Proof. Let $\epsilon > 0$ be given. Let $f : \Omega \times [a, b] \to \mathbb{R}$ be an $L^1(\Omega)$-process. Assume that f is GH-integrable on $[a, b]$. Then f is GH-integrable on every subinterval $[u, v]$ of $[a, b]$. Hence, there exists a positive function δ on $[a, b]$ such that for every δ-fine division $D = \{([t_k, t_{k+1}], \eta_k)\}_{k=1}^m$ of $[a, b]$, we have

$$\mathbf{E}(|(D)\sum_{k=1}^{m} \mathbf{E}(f_{\eta_k}|\mathcal{F}_{t_k})(t_{k+1} - t_k) - F(t_k, t_{k+1})|) \leq \frac{\epsilon}{2}.$$

Let $D' = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ be a δ-fine partial division of $[a, b]$. The set $[a, b] \setminus \cup_{i=1}^n [\, u_i\, ,\ v_i\,]$ consists of a finite number, say p, of disjoint subintervals. Let $K_j, 1 \leq j \leq p$ be the closure of these subintervals respectively. Since K_j is a subinterval of $[a,b]$, there exists a δ-fine division D_j of K_j such that

$$\mathbf{E}(|\tilde{S}(f\mathcal{X}_{K_j},\delta,D_j)-(GH)\int_{K_j} f|)\leq\frac{\epsilon}{2p}.$$

Then $D=D'\cup(\cup_{j=1}^{p}D_j)$ is a δ-fine division of $[a,b]$.

$$\mathbf{E}(|\tilde{S}(f\mathcal{X}_{\cup_{i=1}^{n}[u_i,v_i]},\delta,D')-(GH)\int_{\cup_{i=1}^{n}[u_i,v_i]} f+\sum_{j=1}^{p}(\tilde{S}(f\mathcal{X}_{K_j},\delta,D_j)-(GH)\int_{K_j} f)|)$$

$$=\mathbf{E}(|\tilde{S}(f,\delta,D)-(GH)\int_{[a,b]} f|)\leq\frac{\epsilon}{2}.$$

Hence,

$$\mathbf{E}(|\tilde{S}(f\mathcal{X}_{\cup_{i=1}^{n}[u_i,v_i]},\delta,D')-(GH)\int_{\cup_{i=1}^{n}[u_i,v_i]} f|)$$

$$\leq\frac{\epsilon}{2}+\mathbf{E}(|\sum_{j=1}^{p}(\tilde{S}(f\mathcal{X}_{K_j},\delta,D_i)-(GH)\int_{K_j} f)|)\leq\frac{\epsilon}{2}+p\frac{\epsilon}{2p}=\epsilon.$$

Then,

$$\mathbf{E}(|(D')\sum_{i=1}^{n}\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(v_i-u_i)-F(u_i,v_i)|)\leq\epsilon.$$

□

Similarly, we can prove the following.

Lemma 2.2.10 (Henstock's lemma for square GH-integral)**.** *If f is square GH-integrable on $[a,b]$ and $F(u,v)=(GH)\int_u^v f_t^{[2]}\,dt$ for any $[u,v]\subseteq[a,b]$, then for every $\epsilon>0$, there exists a positive function δ on $[a,b]$ such that for every δ-fine partial division $D'=\{([u_i,v_i],\xi_i)\}$ of $[a,b]$, we have*

$$\mathbf{E}(|(D')\sum_{i=1}^{n}(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2(v_i-u_i)-F(u_i,v_i)|)\leq\epsilon.$$

2.3 Absolute GH-integral

In this section, we shall prove that the absolute GH-integral (induced by Henstock divisions) and the GM-integral (induced by McShane divisions) are equivalent. The ideas of the proofs in this sections are based on that in [3]. However, in the proofs of Theorems 2.3.8 and 2.3.9, a concept in [23] is used.

Definition 2.3.1. Let $[a,b]$ be a given interval. An *elementary set* is a subinterval of $[a,b]$ or a finite number of non-overlapping subintervals of $[a,b]$. We denote by $\mathcal{B}$ the collection of open sets whose complement with respect to $[a,b]$ is an elementary set. We also assume that $[a,b] \in \mathcal{B}$.

Lemma 2.3.2. *Let f GH-integrable on $[a,b]$, with $F(c,d) = (GH)\int_c^d f_t\, dt$ for any $[c,d] \subseteq [a,b]$. Let J be any subset of $[a,b]$. If for any partial division $D = \{([u_i,v_i],\xi_i)\}$ of $[a,b]$, $\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}) \geq 0$, for all i, then for any $\epsilon > 0$, there exists $K_\epsilon \in \mathcal{B}$ with $J \subset K_\epsilon$ such that whenever there exists a finite collection $\{[u_i,v_i]\}_{i=1}^n$ of non-overlapping subintervals of $[a,b]$ with $[u_i,v_i] \subseteq K_\epsilon - J$, for each i, we have*

$$\mathbf{E}(\sum_{i=1}^n F(u_i,v_i)) \leq \epsilon.$$

Proof. In this proof, $(GM)\int_K f$ is denoted by $\int_K f$. Let $J \subseteq [a,b]$ and $\mathcal{B}^* = \{K \in \mathcal{B} : J \subseteq K\}$. Since $[a,b] \in \mathcal{B}^*$, $\mathcal{B}^* \neq \emptyset$. Assume that for any partial division $D = \{([u_i,v_i],\xi_i)\}$ of $[a,b]$, $\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}) \geq 0$, for all i, and f is GH-integrable process on $[a,b]$. Then f is a GH-integrable on $K \in \mathcal{B}^*$. Recall that $K = [a,b]$ or its complement is an elementary set. Let $A = \inf\{\mathbf{E}(\int_K f) : K \in \mathcal{B}^*\}$, which exists and $0 \leq A < \infty$.

Let $\epsilon > 0$ be given. Then there exists $K_\epsilon \in \mathcal{B}^*$ such that

$$\mathbf{E}(\int_{K_\epsilon} f_t\, dt) - A \leq \frac{\epsilon}{2}.$$

Suppose that $\{[u_i, v_i]\}_{i=1}^n$ is a collection of non-overlapping subintervals of $[a, b]$ with $[u_i, v_i] \subseteq K_\epsilon - J$, for each $i = 1, 2, ..., n$. Let K' be the closure of $K_\epsilon - \cup_{i=1}^n [u_i, v_i]$. So $K' \in \mathcal{B}^*$ such that $K' \subseteq K_\epsilon$. Since $\mathbf{E}(f_{\xi_i} | \mathcal{F}_{u_i}) \geq 0$, for all i, then

$$\mathbf{E}(\int_{K'} f_t \; dt) \leq \mathbf{E}(\int_{K_\epsilon} f_t \; dt).$$

We can see that

$$\mathbf{E}(\int_{K'} f_t \; dt) - A \leq \mathbf{E}(\int_{K_\epsilon} f_t \; dt) - A \leq \frac{\epsilon}{2}.$$

Hence,

$$\begin{aligned} \mathbf{E}(\sum_{i=1}^n F(u_i, v_i)) =& \mathbf{E}(\int_{K_\epsilon} f_t \; dt - \int_{K'} f_t \; dt) \\ =& (\mathbf{E}(\int_{K_\epsilon} f_t \; dt) - A) - (\mathbf{E}(\int_{K'} f_t \; dt) - A) \leq \epsilon. \end{aligned}$$

□

Similarly, we can prove the following.

Lemma 2.3.3. *Let f square GH-integrable on $[a, b]$, with $F(c, d) = (GH) \int_c^d f_t^{[2]} \; dt$ for any $[c, d] \subseteq [a, b]$. Let J be any subset of $[a, b]$. Then for any $\epsilon > 0$, there exists $K_\epsilon \in \mathcal{B}$ with $J \subset K_\epsilon$ such that whenever there exists a finite collection $\{[u_i, v_i]\}_{i=1}^n$ of non-overlapping subintervals of $[a, b]$ with $[u_i, v_i] \subseteq K_\epsilon - J$, for each i, we have*

$$\mathbf{E}(\sum_{i=1}^n F(u_i, v_i)) \leq \epsilon.$$

Definition 2.3.4. Let $f : \Omega \times [a, b] \to \mathbb{R}$ be an $L^1(\Omega)$-process. Then f is said to be absolutely GH-integrable on $[a, b]$ if f and $|f|$ are both GH-integrable on $[a, b]$.

By Theorem 2.2.6, we have

Lemma 2.3.5. *If f is absolutely GH-integrable on $[a, b]$, then f is also absolutely GH-integrable on every subinterval $[c, d]$ of $[a, b]$.*

Lemma 2.3.6. *Let $f : \Omega \times [a, b] \to \mathbb{R}$ be an absolutely GH-integrable process on $[a, b]$, with $F(c, d) = (GH)\int_c^d f_t\ dt$ for any $[c, d] \subseteq [a, b]$. Let J be any subset of $[a, b]$. Then for any $\epsilon > 0$, there exists $K_\epsilon \in \mathcal{B}$ with $J \subset K_\epsilon$ such that whenever there exists a finite collection $\{[u_i, v_i]\}_{i=1}^n$ of non-overlapping subintervals of $[a, b]$ with $[u_i, v_i] \subseteq K_\epsilon - J$, for each i, we have*

$$\mathbf{E}(\sum_{i=1}^n |F(u_i, v_i)|) \leq \epsilon \text{ and } \sum_{i=1}^n (v_i - u_i) \leq \epsilon.$$

Proof. Let f be an absolutely GH-integral process. It is easy to see that the constant function $g(\xi) \equiv 1$ is GH-integrable and $(GH)\int_{u_i}^{v_i} g = v_i - u_i$. Moreover

$$\mathbf{E}(\sum_{i=1}^n |\int_{u_i}^{v_i} f_t|) \leq \mathbf{E}(\sum_{i=1}^n \int_{u_i}^{v_i} |f_t|).$$

Hence, by Lemma 2.3.2, we can get the required result. □

Definition 2.3.7 (The generalized McShane integral)**.** Let $f : \Omega \times [a, b] \to \mathbb{R}$ be an $L^1(\Omega)$-process and $F \in L^1(\Omega)$. Then f is said to be *generalized McShane integrable* (or GM-integrable) to F on $[a, b]$ if for every $\epsilon > 0$, there exists a positive function δ defined on $[a, b]$ such that for every δ-fine McShane division $D = \{([t_i, t_{i+1}], \xi_i)\}_{i=1}^n$ of $[a, b]$, we have

$$\mathbf{E}(|\tilde{S}(f, \delta, D) - F|) \leq \epsilon,$$

where

$$\tilde{S}(f, \delta, D) = \sum_{i=1}^n \mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})(t_{i+1} - t_i).$$

We denote F by $(GM)\int_a^b f_t\ dt$, $(GM)\int_a^b f\ dt$ or $(GM)\int_a^b f$.

Moreover, if f is an $L^2(\Omega)$-process then f is said to be square GM-integrable to F on $[a, b]$ if $\tilde{S}$ is replaced by S', where

$$S'(f, \delta, D) = \sum_{i=1}^n [\mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})]^2(t_{i+1} - t_i).$$

We denote F by $(GM)\int_a^b f_t^{[2]}\ dt$, $(GM)\int_a^b f^{[2]}\ dt$ or $(GM)\int_a^b f^{[2]}$.

We remark that Theorems 2.1.2, 2.1.3, 2.2.1, 2.2.3, 2.2.5, 2.2.6 and Lemma 2.2.9 hold for GM-integrable processes. They can be similarly proved.

Next, we shall consider the relation between the GH-integral and the GM-integral.

Theorem 2.3.8. *If f is absolutely GH-integrable on $[a, b]$, then f is GM-integrable on $[a, b]$*

Proof. Although the idea of the proof comes from [3]. However, the concept $S_\epsilon(k)$ comes from [23], since we deal with processes here, whereas in [3], we deal with deterministic functions. Let $f : \Omega \times [a, b] \to \mathbb{R}$ be an $L^1(\Omega)$-process and $\epsilon > 0$ be given. Assume that f is absolutely GH-integrable on $[a, b]$ with $F(u, v) = (GH)\int_u^v f_t\, dt$. There exists a positive function δ with $\delta(\xi) \le 1$ for all $\xi \in [a, b]$, such that for any δ-fine partial division $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$, we have

$$\mathbf{E}(|(D)\sum_{i=1}^{n} \mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(v_i - u_i) - F(u_i, v_i)|) \le \epsilon.$$

Let $\{g_1, g_2, ...\}$ be a countable dense subset of $L^1(\Omega)$. Define

$$S_\epsilon(k) = \{g \in L^1(\Omega) : \mathbf{E}(|g - g_k|) \le \frac{\epsilon}{2}\}.$$

Thus, by definition, $\bigcup_{k=1}^{\infty} S_\epsilon(k) = L^1(\Omega)$.

Let

$$J_\epsilon(1) = \{\xi \in [a, b] : f_\xi \in S_\epsilon(1)\} \text{ and}$$

$$J_\epsilon(k) = \{\xi \in [a, b] : f_\xi \in S_\epsilon(k) \setminus \bigcup_{k=1}^{k-1} S_\epsilon(k)\}, \text{ for each } k \in \{2, 3, ...\}.$$

Note that for any $\xi_1, \xi_2 \in J_\epsilon(k)$, we have

$$\mathbf{E}(|f_{\xi_1} - f_{\xi_2}|) \le \epsilon.$$

Let

$$J_\epsilon(k, n) = \{\xi \in J_\epsilon(k) : 1/(n + 1) < \delta(\xi) \le 1/n\} \text{ for any } n \in \mathbb{N}.$$

Next, we divide $[a, b]$ into $p(n)$ subintervals $[u_q(n), v_q(n)]$, where $q = 1, 2, ..., p(n)$, such that $v_q(n) - u_q(n) \leq 1/(n+1)$.

Let

$$J_\epsilon(k, n, q) = J_\epsilon(k, n) \cap (u_q(n), v_q(n)).$$

From the definition of $J_\epsilon(k, n, q)$, we can see that if $\xi \in J_\epsilon(k, n, q)$, then we have $[u_q(n), v_q(n)] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$.

By Lemma 2.3.6, there exists $K_\epsilon(k, n, q)$ with $J_\epsilon(k, n, q) \subseteq K_\epsilon(k, n, q) \subseteq [u_q(n), v_q(n)]$ such that whenever there exists a finite collection $\{[u_i, v_i]\}_{i=1}^n$ of non-overlapping subintervals of $[u_q(n), v_q(n)]$ with $[u_i, v_i] \subseteq K_\epsilon(k, n, q) - J_\epsilon(k, n, q)$, for each i, we have

$$\mathbf{E}(\sum_{i=1}^{n} |F(u_i, v_i)|) \leq \frac{\epsilon}{(|k|+1)2^{|k|+n+q}}$$

and

$$\sum_{i=1}^{n}(v_i - u_i) \leq \frac{\epsilon}{(|k|+1)2^{|k|+n+q}(\mathbf{E}(g_k)+\epsilon)}.$$

Let B be the set consisting of the end-points of $[u_q(n), v_q(n)]$ for all q. We can define $\delta'(\xi)$ on B in such a way that for any δ'-fine partial McShane division $D = \{([u_i, v_i], \xi_i)\}$, with $\xi_i \in B$, we have

$$\mathbf{E}(|(D)\sum_{\xi_i \in B} \mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(v_i - u_i)|) \leq \frac{\epsilon}{2} \text{ and } \mathbf{E}((D)\sum_{\xi_i \in B} |F(u_i, v_i)|) \leq \frac{\epsilon}{2}, .$$

Then

$$\begin{aligned}
&\mathbf{E}(|(D)\sum_{\xi_i \in B}\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(v_i - u_i) - F(u_i, v_i)|) \\
&\leq \mathbf{E}(|(D)\sum_{\xi_i \in B} \mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(v_i - u_i)|) + \mathbf{E}(|(D)\sum_{\xi_i \in B} F(u_i, v_i)|) \\
&\leq \mathbf{E}(|(D)\sum_{\xi_i \in B} |\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(v_i - u_i)|) + \mathbf{E}((D)\sum_{\xi_i \in B} |F(u_i, v_i)|) \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Now, define $\delta'(\xi)$ on $J_\epsilon(k,n,q)$ such that $0 < \delta'(\xi) \leq \delta(\xi)$, with

$$(\xi - \delta'(\xi), \xi + \delta'(\xi)) \subseteq K_\epsilon(k,n,q) \subseteq [u_q(n), v_q(n)]$$

Let $D' = \{([u_i', v_i'], \xi_i')\}$ be a δ'-fine McShane division, that is, $\xi' \in [u_i', v_i']$ is not required, of $[a,b]$. If $\xi_i' \notin [u_i', v_i']$ and $\xi_i' \in J_\epsilon(k,n,q)$. For the case $[u_i', v_i'] \cap J_\epsilon(k,n,q) = \emptyset$, $[u_i', v_i'] \subseteq K_\epsilon(k,n,q) - J_\epsilon(k,n,q)$. Hence, we have

$$\mathbf{E}(\sum_1 |F(u_i', v_i')|) \leq \epsilon \text{ and } \mathbf{E}(|\sum_1 \mathbf{E}(f_{\xi_i'}|\mathcal{F}_{u_i'})(v_i' - u_i')|) \leq \epsilon,$$

where $\sum_1 = \sum_{[u_i', v_i'] \cap J_\epsilon(k,n,q) = \emptyset}$. Thus, we can see that

$$\begin{aligned}
&\mathbf{E}(|\sum_1 \mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i})(v_i' - u_i') - F(u_i', v_i')|) \\
&\qquad \leq \mathbf{E}(|\sum_1 \mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i})(v_i' - u_i')|) + \mathbf{E}(|\sum_1 F(u_i', v_i')|) \\
&\qquad \leq \mathbf{E}(|\sum_1 \mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i})(v_i' - u_i')|) + \mathbf{E}(\sum_1 |F(u_i', v_i')|) \\
&\qquad \leq 2\epsilon.
\end{aligned}$$

For another case $[u_i', v_i'] \cap J_\epsilon(k,n,q) \neq \emptyset$, there exists $\xi_i'' \in [u_i', v_i'] \cap J_\epsilon(k,n,q)$. Hence we have

$$\xi_i'' \in [u_i', v_i'] \subseteq (\xi_i' - \delta'(\xi_i'), \xi_i' + \delta'(\xi_i')) \subseteq [u_q(n), v_q(n)] \subseteq (\xi_i'' - \delta(\xi_i''), \xi_i'' + \delta(\xi_i'')),$$

then $([u_i', v_i'], \xi_i'')$ is $\delta(\xi'')$-fine and $\xi_i'' \in [u_i', v_i']$.

Let $\sum_2 = \sum_{[u_i', v_i'] \cap J_\epsilon(k,n,q) \neq \emptyset}$. Hence we get

$$
\begin{aligned}
&\mathbf{E}(|\sum_2 \big(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{u_i'})(v_i'-u_i')-F(u_i',v_i')\big)|)\\
&\qquad=\mathbf{E}(|\sum_2 \big((\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{u_i'})-\mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'}))(v_i'-u_i')\\
&\qquad\quad+\mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'})(v_i'-u_i')-F(u_i',v_i')\big)|)\\
&\qquad\le\mathbf{E}(|\sum_2(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{u_i'})-\mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'}))(v_i'-u_i')|)\\
&\qquad\quad+\mathbf{E}\big((|\sum_2\mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'})(v_i'-u_i')-F(u_i',v_i')\big)|)\\
&\qquad=\mathbf{E}(\sum_2(|\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{u_i'})-\mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'}))|(v_i'-u_i'))\\
&\qquad\quad+\mathbf{E}(|\sum_2\big(\mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'})(v_i'-u_i')-F(u_i',v_i')\big)|)\\
&\qquad\le\mathbf{E}(\sum_2(\mathbf{E}(|f_{\xi_i'}-f_{\xi_i''}||\mathcal{F}_{u_i'}))(v_i'-u_i'))\\
&\qquad\quad+\mathbf{E}(|\sum_2\big(\mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'})(v_i'-u_i')-F(u_i',v_i')\big)|)\\
&\qquad\le\sum_2\big(\mathbf{E}(|f_{\xi_i'}-f_{\xi_i''}|)(v_i'-u_i')\\
&\qquad\quad+\mathbf{E}(|\sum_2\mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'})(v_i'-u_i')-F(u_i',v_i'))\big)|\\
&\qquad\le\epsilon\mathbf{E}(\sum_2(v_i'-u_i'))+\mathbf{E}(|\sum_2\mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'})(v_i'-u_i')-F(u_i',v_i'))|\\
&\qquad\le\epsilon(b-a)+\epsilon\ \text{(since f is GH-integrable)}\ .
\end{aligned}
$$

Now, we shall consider the full summation over the division D'

$$
\begin{aligned}
&\mathbf{E}(|(D')\sum\big(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i})(v_i'-u_i')-F(u_i',v_i')\big)|)\\
&\quad\leq \mathbf{E}(|\sum_{\xi_i'\in B}\big(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i})(v_i'-u_i')-F(u_i',v_i')\big)|)\\
&\qquad+\mathbf{E}(|\sum_{\xi_i'\notin B}\big(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i})(v_i'-u_i')-F(u_i',v_i')\big)|)\\
&\quad<\epsilon+\mathbf{E}(|\sum_{\xi_i'\notin B}\big(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i})(v_i'-u_i')-F(u_i',v_i')\big)|)\\
&\quad\leq \epsilon+\mathbf{E}(|\sum_{\xi_i'\in[u_i',v_i']}\big(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i})(v_i'-u_i')-F(u_i',v_i')\big)|)\\
&\qquad+\mathbf{E}(|\sum_{\xi_i'\notin[u_i',v_i']}\big(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i})(v_i'-u_i')-F(u_i',v_i')\big)|)\\
&\quad\leq \epsilon+\epsilon+\mathbf{E}(|\sum_{1}\big(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i})(v_i'-u_i')-F(u_i',v_i')\big)|)\\
&\qquad+\mathbf{E}(|\sum_{2}\big(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i})(v_i'-u_i')-F(u_i',v_i')\big)|)\ \text{(since f is GH-integrable)}\\
&\quad\leq \epsilon+\epsilon+\epsilon+\epsilon+\epsilon(b-a)+\epsilon=(5+b-a)\epsilon.
\end{aligned}
$$

Therefore f is GM-integrable on $[a,b]$. □

Theorem 2.3.9. *Let $f:\Omega\times[a,b]\to\mathbb{R}$ be an $L^2(\Omega)$-process. Then f is square GH-integrable on $[a,b]$ if and only if f is square GM-integrable on $[a,b]$*

Proof. The following proof is similar to that of Theorem 2.3.8. If f is square GM-integrable then it is clear that f is square GH-integrable. We shall prove the converse now. Let $f:\Omega\times[a,b]\to\mathbb{R}$ be an $L^2(\Omega)$-process and $\epsilon>0$ be given. Assume that f is square GH-integrable on $[a,b]$ with $F(u,v)=(GH)\int_u^v f^{[2]}$. Then there exists a positive function δ with $\delta(\xi)\leq 1$ for $\xi\in[a,b]$, such that for any δ-fine partial division $D=\{([u_i,v_i],\xi_i)\}_{i=1}^n$, we have

$$\mathbf{E}(|(D)\sum_{i=1}^n(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2(v_i-u_i)-F(u_i,v_i)|)\leq\epsilon.$$

Let $\{g_1,g_2,...\}$ be a countable dense subset of $L^2(\Omega)$. For each k, define

$$S_\epsilon(k)=\{g\in L^2(\Omega):\mathbf{E}(|g-g_k|^2)\leq\frac{\epsilon}{4}\}.$$

Thus, by definition, $\bigcup_{k=1}^{\infty} S_\epsilon(k) = L^2(\Omega)$.

Let

$$J_\epsilon(1) = \{\xi \in [a,b] : f_\xi \in S_\epsilon(1)\}.$$

$$J_\epsilon(k) = \{\xi \in [a,b] : f_\xi \in S_\epsilon(k) \setminus \bigcup_{k=1}^{k-1} S_\epsilon(k)\} \text{ for each } k \in \{2,3,...\}.$$

Note that for any $\xi_1, \xi_2 \ \in \ J_\epsilon(k)$ we have

$$\mathbf{E}(|f_{\xi'} - f_{\xi''}|^2) \leq \epsilon.$$

Let

$$J_\epsilon(k,n) = \{\xi \in J_\epsilon(k) : 1/(n+1) < \delta(\xi) \leq 1/n\} \text{ for any } n \in \mathbb{N}.$$

Next, we divide $[a,b]$ into $p(n)$ subintervals $[u_q(n), v_q(n)]$, where $q = 1, 2, ..., p(n)$, such that $v_q(n) - u_q(n) \leq 1/(n+1)$.

Let

$$J_\epsilon(k,n,q) = J_\epsilon(k,n) \cap (u_q(n), v_q(n)).$$

From the definition of $J_\epsilon(k,n,q)$, we can see that if $\xi \in J_\epsilon(k,n,q)$, then we have $[u_q(n), v_q(n)] \subseteq (\xi - \delta(\xi), \xi + \delta(\xi))$.

By Lemma 2.3.6, there exists $K_\epsilon(k,n,q)$ with $J_\epsilon(k,n,q) \subseteq K_\epsilon(k,n,q) \subseteq [u_q(n), v_q(n)]$ such that whenever there exists a finite collection $\{[u_i, v_i]\}_{i=1}^n$ of non-overlapping subintervals of $[u_q(n), v_q(n)]$ with $[u_i, v_i] \subseteq K_\epsilon(k,n,q) - J_\epsilon(k,n,q)$ for each i, we have

$$\mathbf{E}(\sum_{i=1}^{n} |F(u_i, v_i)|) <\leq \frac{\epsilon}{(|k|+1)2^{|k|+n+q}}$$

and

$$\sum_{i=1}^{n} (v_i - u_i) \leq \frac{\epsilon}{(|k|+1)2^{|k|+n+q} 2(\mathbf{E}(g_k^2) + \epsilon)}.$$

Let B be the set consisting of the end-points of $[u_q(n), v_q(n)]$ for all q. We can define $\delta'(\xi)$ on B in such a way that for any δ'-fine partial McShane division

$D = \{([u_i, v_i], \xi_i)\}$, with $\xi_i \in B$, we have

$$\mathbf{E}(|(D)\sum_{\xi_i \in B}(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2(v_i - u_i)|) \leq \frac{\epsilon}{2} \text{ and } \mathbf{E}((D)\sum_{\xi_i \in B}|F(u_i, v_i)|) \leq \frac{\epsilon}{2}.$$

Then

$$\begin{aligned}
&\mathbf{E}(|(D)\sum_{\xi_i \in B}(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2(v_i - u_i) - F(u_i, v_i)|) \\
&\qquad \leq \mathbf{E}(|(D)\sum_{\xi_i \in B}(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2(v_i - u_i)|) + \mathbf{E}((D)\sum_{\xi_i \in B}|F(u_i, v_i)|) \\
&\qquad \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Now, define $\delta'(\xi)$ on $J_\epsilon(k, n, q)$ such that $0 < \delta'(\xi) < \min\{\delta(\xi), \frac{1}{2^{k+1}(\epsilon + \mathbf{E}(f_k^2))^{1/2}}\}$, with

$$(\xi - \delta'(\xi), \xi + \delta'(\xi)) \subseteq K_\epsilon(k, n, q) \subset [u_q(n), v_q(n)]$$

Let $D' = \{([u'_i, v'_i], \xi'_i)\}$ be a δ'-fine McShane division, that is, $\xi' \in [u'_i, v'_i]$ is not required, of $[a, b]$. If $\xi'_i \notin [u'_i, v'_i]$ and $\xi' \in J_\epsilon(k, n, q)$. For the case $[u'_i, v'_i] \cap J_\epsilon(k, n, q) = \emptyset$, $[u'_i, v'_i] \subseteq K_\epsilon(k, n, q) - J_\epsilon(k, n, q)$. Hence, we have

$$\mathbf{E}(|\sum_1 F(u'_i, v'_i)|) \leq \epsilon \text{ and } \mathbf{E}(|\sum_1(\mathbf{E}(f_{\xi'_i}|\mathcal{F}_{u'_i}))^2(v'_i - u'_i)|) \leq \epsilon,$$

where $\sum_1 = \sum_{[u'_i, v'_i] \cap J_\epsilon(k,n,q) = \emptyset}$. Thus, we can see that

$$\begin{aligned}
&\mathbf{E}(|\sum_1(\mathbf{E}(f_{\xi'_i}|\mathcal{F}_{t_i}))^2(v'_i - u'_i) - F(u'_i, v'_i)|) \\
&\qquad \leq \mathbf{E}(|\sum_1(\mathbf{E}(f_{\xi'_i}|\mathcal{F}_{t_i}))^2(v'_i - u'_i) - F(u'_i, v'_i)|) \\
&\qquad \leq \mathbf{E}(|\sum_1(\mathbf{E}(f_{\xi'_i}|\mathcal{F}_{t_i}))^2(v'_i - u'_i)|) + \mathbf{E}(|\sum_1 F(u'_i, v'_i)|) \\
&\qquad \leq 2\epsilon.
\end{aligned}$$

For another case $[u'_i, v'_i] \cap J_\epsilon(k, n, q) \neq \emptyset$, there exists $\xi''_i \in [u'_i, v'_i] \cap J_\epsilon(k, n, q)$. Hence we have

$$\xi''_i \in [u'_i, v'_i] \subseteq (\xi'_i - \delta'(\xi'_i), \xi'_i + \delta'(\xi'_i)) \subseteq [u_q(n), v_q(n)] \subseteq (\xi''_i - \delta(\xi''_i), \xi''_i + \delta(\xi''_i)),$$

then $([u_i', v_i'], \xi_i'')$ is $\delta(\xi'')$-fine and $\xi_i'' \in [u_i', v_i']$.

Let $\sum_2 = \sum_{[u_i',v_i'] \cap J_\epsilon(k,n,q) \neq \emptyset}$. Hence we get

$$
\begin{aligned}
&\mathbf{E}(|\sum_2 \big((\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{u_i'}))^2 (v_i' - u_i') - F(u_i', v_i')\big)|)\\
=&\mathbf{E}(|\sum_2 \big(((\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{u_i'}))^2 - (\mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'}))^2)(v_i' - u_i')\\
&+ (\mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'}))^2 (v_i' - u_i') - F(u_i', v_i')|)\big)\\
\leq&\mathbf{E}(|\sum_2 \big(((\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{u_i'}))^2 - (\mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'}))^2)(v_i' - u_i')\big)|)\\
&+ \mathbf{E}(|\sum_2 \big((\mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'}))^2 (v_i' - u_i') - F(u_i', v_i')\big)|)\\
\leq&\sum_2 \mathbf{E}(|(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{u_i'}))^2 - (\mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'}))^2|)(v_i' - u_i')\\
&+ \mathbf{E}(|\sum_2 \big((\mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'}))^2 (v_i' - u_i') - F(u_i', v_i')\big)|)\\
\leq&\sum_2 \mathbf{E}(|\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{u_i'}) - \mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'})||\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{u_i'}) + \mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'})|(v_i' - u_i'))\\
&+ \epsilon \text{ (since } f \text{ is square GH-integrable)}\\
\leq&\sum_2 \big(\mathbf{E}(|\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{u_i'}) - \mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'})|^2)\big)^{1/2} \big(\mathbf{E}(|\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{u_i'}) + \mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'})|^2)\big)^{1/2} (v_i' - u_i') + \epsilon\\
\leq&\sum_2 (\mathbf{E}(|f_{\xi'} - f_{\xi''}|^2))^{1/2} (\mathbf{E}(|\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{u_i'}) + \mathbf{E}(f_{\xi_i''}|\mathcal{F}_{u_i'})|^2))^{1/2} (v_i' - u_i') + \epsilon\\
\leq&\sqrt{\epsilon} \sum_2 (4(\epsilon + \mathbf{E}(f_k^2)))^{1/2} \frac{1}{2^{k+1}(\epsilon + \mathbf{E}(f_k^2))^{1/2}} + \epsilon\\
\leq&\sqrt{\epsilon} \sum_2 \frac{1}{2^k} + \epsilon\\
\leq&\sqrt{\epsilon} + \epsilon.
\end{aligned}
$$

Now, we shall consider the full summation of division D'.

$$
\begin{aligned}
&\mathbf{E}(|(D')\sum(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i}))^2(v_i'-u_i')-F(u_i',v_i')|)\\
&\leq \mathbf{E}(|\sum_{\xi_i'\in B}(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i}))^2(v_i'-u_i')-F(u_i',v_i')|)\\
&\quad+\mathbf{E}(|\sum_{\xi_i'\notin B}(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i}))^2(v_i'-u_i')-F(u_i',v_i')|)\\
&<\epsilon+\mathbf{E}(|\sum_{\xi_i'\notin B}(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i}))^2(v_i'-u_i')-F(u_i',v_i')|)\\
&\leq \epsilon+\mathbf{E}(|\sum_{\xi_i'\in[u_i',v_i']}(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i}))^2(v_i'-u_i')-F(u_i',v_i')|)\\
&\quad+\mathbf{E}(|\sum_{\xi_i'\notin[u_i',v_i']}(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i}))^2(v_i'-u_i')-F(u_i',v_i')|)\\
&\leq \epsilon+\epsilon+\mathbf{E}(|\sum_{1}(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i}))^2(v_i'-u_i')-F(u_i',v_i')|)\\
&\quad+\mathbf{E}(|\sum_{2}(\mathbf{E}(f_{\xi_i'}|\mathcal{F}_{t_i}))^2(v_i'-u_i')-F(u_i',v_i')|)\ \text{(since } f\text{is square GH-integrable)}\\
&\leq \epsilon+\epsilon+\epsilon+\epsilon+\sqrt{\epsilon}+\epsilon.
\end{aligned}
$$

Therefore f is square GM-integrable on $[a,b]$. □

2.4 Absolute continuity

It is well-known that absolute continuity of the Lebesgue integral plays an important role in the theory of Lebesgue integration. In this section, we shall prove that the square GM-integral also has this property, which will be used in the next section. The idea of the proof of Theorem 2.4.2 in this section comes from [16].

Definition 2.4.1. A stochastic process $\mathbf{X}:\Omega\times[a,b]\to\mathbb{R}$ is said to be $AC([a,b])$ if for every $\epsilon>0$, there exists a positive real number η such that for any partial partition $D=\{([x_i,y_i]\}_{i=1}^n$ of $[a,b]$, with $(D)\sum_{i=1}^n(y_i-x_i)\leq\eta$, we have

$$|\sum_{i=1}^n\mathbf{E}[\mathbf{X}_{y_i}-\mathbf{X}_{x_i}]|\leq\epsilon.$$

Theorem 2.4.2. *If f is square GM-integrable to F on $[a, b]$, then F has the $AC([a, b])$ property.*

Proof. Let f be square GM-integrable to F and $\epsilon > 0$ be given. We remark that Lemma 2.2.9 also holds for square GM-integral. Then there exists a positive function δ on $[a, b]$ such that for every δ-fine partial McShane division $D = \{([u_i, v_i], \xi_i)\}$ we have

$$\mathbf{E}(|(D)\sum(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2(v_i - u_i) - F(u_i, v_i)|) \leq \frac{\epsilon}{2}.$$

Hence,

$$\begin{aligned}|(D)\sum \mathbf{E}(F(u_i, v_i))| \leq& |(D)\sum \mathbf{E}((\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2(v_i - u_i))| + \frac{\epsilon}{2} \\ \leq& (D)\sum \mathbf{E}(f^2_{\xi_i})(v_i - u_i) + \frac{\epsilon}{2}.\end{aligned}$$

Let $D' = \{([t_i, t_{i+1}], \xi_i)\}_{i=1}^n$ be a fixed δ-fine McShane division of $[a, b]$ and $M > 0$ with

$$M \geq \max_{1\leq i\leq n}\{\mathbf{E}(f^2_{\xi_i})\}.$$

Choose $\eta \leq \frac{\epsilon}{2M}$. Let $\{[x_i, y_i]\}$ be a partial partition of $[a, b]$. Assume that $\sum |y_j - x_j| \leq \eta$. Let the refinement of $\{[x_i, y_i]\}$ and $\{[t_i, t_{i+1}]\}$ on $\cup_i[x_i, y_i]$ be $\{[a_k, b_k]\}_{k=1}^q$. Then $\cup_k[a_k, b_k] = \cup_i[x_i, y_i]$. If $[a_k, b_k]$ is a subinterval of $[t_i, t_{i+1}]$, then we choose ξ_i as an associate point of $[a_k, b_k]$, denote ξ_i by η_k. From this construction we get new δ-fine partial McShane division $D'' = \{([a_k, b_k], \eta_k)\}_{k=1}^q$. Hence, we can see that

$$\begin{aligned}|\sum \mathbf{E}(F(x_j, y_j))| =& |(D'')\sum_{k=1}^q \mathbf{E}(F(a_k, b_k))| \\ \leq& (D'')\sum_{k=1}^q \mathbf{E}(f^2_{\eta_k})(b_k - a_k) + \frac{\epsilon}{2} \\ \leq& M(D'')\sum_{k=1}^q (b_k - a_k) + \frac{\epsilon}{2} \\ \leq& M\frac{\epsilon}{2M} + \frac{\epsilon}{2} \\ =& \epsilon.\end{aligned}$$

Therefore, F is an $AC([a,b])$ process. □

Corollary 2.4.3. *If f is a square GM-integrable process on $[a,b]$, then for every $\epsilon > 0$, there exist a positive function δ on $[a,b]$ and a positive real number η such that for any δ-fine partial McShane division $D = \{([u_i, v_i], \xi_i)\}$ of $[a,b]$ with $(D)\sum(v_i - u_i) \leq \eta$, we have*

$$\mathbf{E}[(D)\sum(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2(v_i - u_i)] \leq \epsilon$$

Proof. The proof is standard in the theory of Henstock integration. Let f be a square GM-integrable process and $\epsilon > 0$ be given. From Theorem 2.4.2, there exists a positive real number η such that for any partial partition $D = \{[u_i, v_i]\}$ of $[a,b]$ with $(D)\sum|v_i - u_i| \leq \eta$, we have

$$|\sum_{i=1}^{n} \mathbf{E}(F(u_i, v_i))| \leq \frac{\epsilon}{2}.$$

From Henstock's Lemma, there exists a positive function δ on $[a,b]$ such that for any δ-fine partial McShane partition $D = \{([u_i, v_i], \xi_i)\}_{i=1}^{n}$ we have

$$\mathbf{E}(|(D)\sum_{i=1}^{n}(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2(v_i - u_i) - F(u_i, v_i)|) \leq \frac{\epsilon}{2}.$$

Hence, we can see that

$$\begin{aligned}
\mathbf{E}[\sum_{i=1}^{n}(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2(v_i - u_i)] =& \mathbf{E}[\sum_{i=1}^{n}(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2(v_i - u_i) - F(u_i, v_i)] \\
&+ \mathbf{E}[\sum_{i=1}^{n}(F(u_i, v_i))] \\
\leq& \mathbf{E}(|(D)\sum_{i=1}^{n}(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2(v_i - u_i) - F(u_i, v_i)|) \\
&+ |\sum_{i=1}^{n}\mathbf{E}(F(u_i, v_i))| \\
\leq& \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

□

2.5 Integrable processes

In this section, we shall prove that if f is an adapted $L^2(\Omega)$-process, then f is square GM-integrable if and only if f^2 is Bochner integrable. This is our main result in this chapter. For the definition of Bochner integral, see [17] or Chapter 1 Section 1.5. First, we shall state some known results without proofs.

Definition 2.5.1. [8, 9] Let $f : \Omega \times [a, b] \to \mathbb{R}$ be an $L^1(\Omega)$-process. Then f is said to be M-integrable on $[a, b]$ if for each $\epsilon > 0$, there exists a positive function δ on $[a, b]$ such that whenever $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ and $D' = \{([u_i, v_i], \eta_i)\}_{i=1}^n$ are two δ-fine McShane divisions of $[a, b]$ (of the forms as above), we have

$$\sum_{i=1}^{n} \mathbf{E}(|(f_{\xi_i} - f_{\eta_i})(v_i - u_i)|) \leq \epsilon.$$

Theorem 2.5.2. *[8, 9] Let $f : \Omega \times [a, b] \to \mathbb{R}$ be an $L^1(\Omega)$-process. Then f is M-integrable on $[a, b]$ if and only if f is Bochner integrable on $[a, b]$. Their integrals are equal.*

Theorem 2.5.3. *[8, 9] Let $f : \Omega \times [a, b] \to \mathbb{R}$ be an $L^1(\Omega)$-process. Then f is M-integrable on $[a, b]$ if and only if $\mathbf{E}(|f_t|)$ is Lebesgue integrable on $[a, b]$. Their integrals are equal.*

The integral of f on $[a, b]$ is denoted by $(L) \int_a^b \mathbf{E}(|f_t|)\, dt$.

Definition 2.5.4. Let δ be a positive function on $[a, b]$. Then a partial division $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a, b]$ is called a δ-fine partial belated division if $[u_i, v_i] \subset [\xi_i, \xi_i + \delta(\xi_i))$, for each i and $\{[u_i, v_i]\}_{i=1}^n$ are non-overlapping intervals.

We remark that a δ-fine partial belated division is a δ-fine partial McShane division.

Definition 2.5.5. [14, 15] Let $g : [a, b] \to \mathbb{R}$. Then g is said to be belated integrable to $A \in \mathbb{R}$ on $[a, b]$ if for every $\epsilon > 0$, there exist $\eta > 0$ and a positive function δ on

$[a,b]$ such that for any δ-fine partial belated division $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a,b]$ with $\left| b - a - \sum_{i=1}^n |v_i - u_i| \right| \leq \eta$, we have

$$|\sum_{i=1}^n g(\xi_i)(v_i - u_i) - A| \leq \epsilon.$$

Theorem 2.5.6. *[14] Let $g : [a,b] \to \mathbb{R}$. Then g is belated integrable on $[a,b]$ if and only if g is Lebesgue integrable on $[a,b]$.*

Now we shall prove the following new results.

Theorem 2.5.7. *Let $f : \Omega \times [a,b] \to \mathbb{R}$ be an adapted $L^2(\Omega)$-process. Then f is square GM-integrable on $[a,b]$ if and only if f^2 is M-integrable (Bochner) on $[a,b]$.*

Proof. Suppose f is square GM-integrable on $[a,b]$. For each $\epsilon > 0$, there exists a positive function δ on $[a,b]$ such that whenever $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ is a δ-fine McShane division of $[a,b]$, we have

$$\mathbf{E}\left(\left| \sum_{i=1}^n \left(\mathbf{E}(f_{\xi_i} | \mathcal{F}_{u_i}) \right)^2 (v_i - u_i) - (GM) \int_a^b f \right| \right) \leq \epsilon.$$

On the other hand, from Corollary 2.4.3, there exists $\eta > 0$ such that when $D' = \{([x_i, y_i])\}_{i=1}^m$ is a δ-fine partial McShane division of $[a,b]$ with $\sum_{i=1}^m |x_i - y_i| \leq \eta$, we have

$$\mathbf{E}\left(\left| \sum_{i=1}^m \left(\mathbf{E}(f_{\xi_i} | \mathcal{F}_{x_i}) \right)^2 (y_i - x_i) \right| \right) \leq \epsilon.$$

Let $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ be a δ-fine partial belated division of $[a,b]$ and $D' = \{([x_i, y_i], \eta_i)\}_{i=1}^m$ be a δ-fine partial McShane division of the closure of $[a,b] \setminus \cup_{i=1}^n [u_i, v_i]$ such that $\sum_{i=1}^m |x_i - y_i| \leq \eta$. Then

$$\mathbf{E}\left(\left| (D) \sum \left(\mathbf{E}(f_{\xi_i} | \mathcal{F}_{u_i}) \right)^2 (v_i - u_i) - (GM) \int_a^b f \right| \right) \leq 2\epsilon.$$

Observe that f is adapted, so f_{ξ_i} is $\mathcal{F}_{\xi_i}$-measurable. Then f_{ξ_i} is $\mathcal{F}_{u_i}$-measurable. Consequently $\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}) = f_{\xi_i}$. Therefore

$$|(D)\sum \mathbf{E}(f^2_{\xi_i})(v_i - u_i) - \mathbf{E}((GM)\int_a^b f)| \leq 2\epsilon.$$

By Theorems 2.5.6 and 2.5.3, f^2 is M-integrable on $[a,b]$

Conversely, suppose f^2 is M-integrable on $[a,b]$. Then $|f|^2$ is M-integrable on $[a,b]$. Observe that $f^+ = \{f^+_t : t \in [a,b]\} = \frac{|f|+f}{2}$ and $f^- = f - f^+$. Thus $(f^+)^2$ and $(f^-)^2$ are M-integrable on $[a,b]$. Therefore, in the following proof, we may assume that f is nonnegative. Suppose f^2 is M-integrable on $[a,b]$. Let $D = \{([u_i,v_i],\xi_i)\}$ and $D' = \{([u_i,v_i],\eta_i)\}$ be fixed. Denote $\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})$ and $\mathbf{E}(f_{\eta_i}|\mathcal{F}_{u_i})$ by s_i and t_i respectively.

Then

$$\begin{aligned}
&\mathbf{E}\Big(\big|(D)\textstyle\sum s_i^2(v_i-u_i) - (D')\sum t_i^2(v_i-u_i)\big|\Big)\\
\leq&\mathbf{E}\Big(\big|\sum_i (s_i^2 - t_i^2)(v_i-u_i)\big|\Big)\\
\leq&\mathbf{E}\Big(\sum_i \big|s_i - t_i\big|\big|s_i + t_i\big|(v_i-u_i)\Big)\\
\leq&\Big(\mathbf{E}(\sum_i \big|s_i - t_i\big|^2(v_i-u_i))\Big)^{\frac{1}{2}}\Big(\mathbf{E}(\sum_i \big|s_i + t_i\big|^2(v_i-u_i))\Big)^{\frac{1}{2}}\\
\leq 2&\Big(\mathbf{E}(\sum_i \big|s_i - t_i\big|^2(v_i-u_i))\Big)^{\frac{1}{2}}\Big[\Big(\mathbf{E}(\sum_i \big|s_i\big|^2(v_i-u_i))\Big)^{\frac{1}{2}} + \Big(\mathbf{E}(\sum_i \big|t_i\big|^2(v_i-u_i))\Big)^{\frac{1}{2}}\Big]\\
\leq 2&\Big(\mathbf{E}(\sum_i (f_{\xi_i} - f_{\eta_i})^2(v_i-u_i))\Big)^{\frac{1}{2}}\Big[\Big(\mathbf{E}(\sum_i f^2_{\xi_i}(v_i-u_i))\Big)^{\frac{1}{2}} + \Big(\mathbf{E}(\sum_i f^2_{\eta_i}(v_i-u_i))\Big)^{\frac{1}{2}}\Big]\\
\leq 2&\Big(\mathbf{E}(\sum_i (\Big|\big|f_{\xi_i}\big|^2 - \big|f_{\eta_i}\big|^2\Big|)(v_i-u_i))\Big)^{\frac{1}{2}}\Big[\Big(\mathbf{E}(\sum_i f^2_{\xi_i}(v_i-u_i))\Big)^{\frac{1}{2}} + \Big(\mathbf{E}(\sum_i f^2_{\eta_i}(v_i-u_i))\Big)^{\frac{1}{2}}\Big].
\end{aligned}$$

Now, suppose f^2 is M-integrable on $[a,b]$. Then for each $\epsilon > 0$, we can find a positive function δ on $[a,b]$ such that for any two δ-fine McShane divisions of $[a,b]$, in the above, first term on the right hand side is less than ϵ and the second term is bounded. Thus f is square GM-integral on $[a, b]$. Note that in the last inequality we use the fact that $|x - y|^2 \leq |x^2 - y^2|$ if $x, y \geq 0$. $\square$

Chapter 3

The Generalized Itô Integral

In this chapter, we shall define an integral of processes with respect to Brownian motion, without assuming adaptedness. This integral is called the GI-integral, which has properties similar to that of the classical Itô integral, see Section 3.3. Furthermore, we shall prove that if a process is adapted, then the GI-integral and the classical Itô integral are equivalent.

3.1 Definition of the GI-integral

In this section, first we shall define the generalized Itô integral. Then we shall prove two standard results, namely Theorems 3.1.2 and 3.1.6. Theirs proofs are standard. Finally, we shall give three examples.

Definition 3.1.1 (The generalized Itô integral)**.** Let $f : \Omega \times [a, b] \to \mathbb{R}$ be an $L^2(\Omega)$-process, $B : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}$ be a standard Brownian motion, and $F \in L^2(\Omega)$. Then f is said to be *generalized Itô integrable* (or GI-integrable) to F on $[a, b]$ if for every $\epsilon > 0$, there exists a positive function δ defined on $[a, b]$ such that for every δ-fine McShane division $D = \{([t_i, t_{i+1}], \xi_i)\}_{i=1}^n$ of $[a, b]$, we have

$$\mathbf{E}[S(f, \delta, D) - F]^2 \leq \epsilon,$$

where

$$S(f, \delta, D) = \sum_{i=1}^{n} \mathbf{E}(f_{\xi_i} | \mathcal{F}_{t_i})(B_{t_{i+1}} - B_{t_i}).$$

We denote F by $(GI)\int_a^b f_t \, dB_t$ or $(GI)\int_a^b f \, dB$.

Theorem 3.1.2. *The integral F in Definition 3.1.1 is unique up to a set of $\mathbf{P}$-measure zero.*

Proof. The proof is standard. Let $\epsilon > 0$. Assume that F_1 and F_2 satisfy the conditions in Definition 3.1.1, i.e., there exist positive function δ_i on $[a, b]$ such that for every δ_i-fine McShane divisions D_i, where $i = 1, 2$, of $[a, b]$, we have

$$\mathbf{E}[|S(f, \delta_1, D_1) - F_1|^2] \le \frac{\epsilon}{4},$$

and

$$\mathbf{E}[|S(f, \delta_2, D_2) - F_2|^2] \le \frac{\epsilon}{4}.$$

Pick $\delta = \min\{\delta_1, \delta_2\}$, then a δ-fine McShane division of $[a, b]$ is also a δ_1-fine McShane division and a δ_2-fine McShane division of $[a, b]$. Then

$$\begin{aligned}
\mathbf{E}[|F_2 - F_1|^2] =& \mathbf{E}[|(S(f, \delta, D) - F_1) - (S(f, \delta, D) - F_2)|^2] \\
\le& \mathbf{E}[(|S(f, \delta, D) - F_1| + |S(f, \delta, D) - F_2|)^2] \\
\le& \mathbf{E}[2|S(f, \delta, D) - F_1|^2 + 2|S(f, \delta, D) - F_2|^2] \\
=& 2\mathbf{E}[|S(f, \delta, D) - F_1|^2] + 2\mathbf{E}[|S(f, \delta, D) - F_1|^2] \\
=& 2(\frac{\epsilon}{4}) + 2(\frac{\epsilon}{4}) = \epsilon.
\end{aligned}$$

Hence, $\mathbf{E}[|F_1 - F_2|^2] = 0$. We can conclude that $F_1 = F_2$ almost surely, i.e., except only a set of $\mathbf{P}$-measure zero. □

Example 3.1.3. *(i) Let $h : \Omega \to \mathbb{R}$ be a bounded random variable on $(\Omega, \mathcal{F}, \mathbf{P})$, i.e., there exists $M \ge 0$ such that $|h(\omega)| \le M$ for all $\omega \in \Omega$. Let $s \in [a, b]$ be fixed* *and* $f : \Omega \times [a, b] \to \mathbb{R}$ *be a process, defined by*

$$f_t(\omega) = \begin{cases} h(\omega) & , \text{ if } t = s; \\ 0 & , \text{ if } t \neq s, \end{cases}$$

for all $\omega \in \Omega$. *Then* f *is GI-integrable to zero on* $[a, b]$.

Proof. Let $\epsilon > 0$ be given. Choose $\delta(\xi) = \epsilon/2(M+1)^2$ for all $\xi \in [a, b]$. For any δ-fine McShane division $D = \{([t_i, t_{i+1}], \xi_i)\}_{i=1}^n$ of $[a, b]$, if $\xi_i \neq s$ for all $i = 1, 2, ..., n$, then $f_{\xi_i} \equiv 0$. So, let us consider the following case, there exists $\xi_k = s$ for some $k \in \{1, 2, ..., n\}$. We can see that

$$\begin{aligned} \mathbf{E}[S(f, \delta, D) - 0]^2 =& \mathbf{E}[\sum_{i=1}^{n} \mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})(B_{t_{i+1}} - B_{t_i})]^2 \\ =& \mathbf{E}[\mathbf{E}(f_s|\mathcal{F}_{t_k})(B_{t_{k+1}} - B_{t_k})]^2 \\ \leq& \mathbf{E}[\mathbf{E}(M|\mathcal{F}_{t_k})(B_{t_{k+1}} - B_{t_k})]^2 \\ =& \mathbf{E}[M(B_{t_{k+1}} - B_{t_k})]^2 \\ =& M^2\mathbf{E}[B_{t_{k+1}} - B_{t_k}]^2 \\ =& M^2(t_{k+1} - t_k) \\ \leq& M^2(\frac{\epsilon}{(M+1)^2}) \\ \leq& \epsilon. \end{aligned}$$

Hence, f is GI-integrable to zero on $[a, b]$. $\square$

(ii) The result of (i) still holds with boundedness of h replaced by $\mathbf{E}(h^2) < \infty$. It can be proved by observing that

$$\begin{aligned} \mathbf{E}((\mathbf{E}(f_s|\mathcal{F}_{t_k}))^2(B_{t_{k+1}} - B_{t_k})^2) =& \mathbf{E}[\mathbf{E}[(\mathbf{E}(f_s|\mathcal{F}_{t_k}))^2(B_{t_{k+1}} - B_{t_k})^2|\mathcal{F}_{t_k}]] \\ =& \mathbf{E}((\mathbf{E}(f_s|\mathcal{F}_{t_k}))^2(t_{k+1} - t_k)) \\ \leq& \mathbf{E}(f_s^2)(t_{k+1} - t_k) \\ =& \mathbf{E}(h^2)(t_{k+1} - t_k). \end{aligned}$$

Example 3.1.4. *The standard Brownian motion $B : \Omega \times [0,T] \to \mathbb{R}$ is GI-integrable with respect to itself, and*

$$(GI)\int_0^T B_t \; dB_t = \frac{1}{2}B_T^2 - \frac{1}{2}T.$$

Proof. Let $\epsilon > 0$ be given. Choose $\delta(\xi) = \epsilon/2(T+1)$, for every δ-fine McShane division $D = \{([t_i, t_{i+1}], \xi_i)\}_{i=1}^n$ of $[0,T]$, we have

$$\mathbf{E}[(D)\sum_{i<j}\{(B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1} - t_i)\}\{(B_{t_{j+1}} - B_{t_j})^2 - (t_{j+1} - t_j)\}]$$

$$\begin{aligned}
=&\mathbf{E}[\mathbf{E}[(D)\sum_{i<j}\{(B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1} - t_i)\}\{(B_{t_{j+1}} - B_{t_j})^2 - (t_{j+1} - t_j)\}|\mathcal{F}_{t_j}]] \\
=&\mathbf{E}[(D)\sum_{i<j}\{(B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1} - t_i)\}\{\mathbf{E}[(B_{t_{j+1}} - B_{t_j})^2|\mathcal{F}_{t_j}] - (t_{j+1} - t_j)\}] \\
=&\mathbf{E}[(D)\sum_{i<j}\{(B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1} - t_i)\}\{(t_{j+1} - t_j) - (t_{j+1} - t_j)\}] = 0.
\end{aligned}$$

Thus

$\mathbf{E}[S(B,\delta,D)-\frac{1}{2}(B_T^2-T)]^2$

$$
\begin{aligned}
&=\mathbf{E}[(D)\sum_{i=1}^{n}\mathbf{E}(B_{\xi_i}|\mathcal{F}_{t_i})(B_{t_{i+1}}-B_{t_i})-\frac{1}{2}(B_T^2-T)]^2\\
&=\mathbf{E}[(D)\sum_{i=1}^{n}\{B_{t_i}(B_{t_{i+1}}-B_{t_i})-\frac{1}{2}(B_{t_i+1}^2-B_{t_i}^2)+\frac{1}{2}(t_{i+1}-t_i)\}]^2\\
&=\mathbf{E}[(-\frac{1}{2})(D)\sum_{i=1}^{n}\{(B_{t_{i+1}}-B_{t_i})^2-(t_{i+1}-t_i)\}]^2\\
&=\frac{1}{4}\mathbf{E}[(D)\sum_{i=1}^{n}\{(B_{t_{i+1}}-B_{t_i})^2-(t_{i+1}-t_i)\}]^2\\
&=\frac{1}{4}\mathbf{E}[(D)\sum_{i=1}^{n}\{(B_{t_{i+1}}-B_{t_i})^2-(t_{i+1}-t_i)\}^2]\\
&\quad+\frac{1}{2}\mathbf{E}[(D)\sum_{i<j}\{(B_{t_{i+1}}-B_{t_i})^2-(t_{i+1}-t_i)\}\{(B_{t_{j+1}}-B_{t_j})^2-(t_{j+1}-t_j)\}]\\
&=\frac{1}{4}[(D)\sum_{i=1}^{n}\mathbf{E}[(B_{t_{i+1}}-B_{t_i})^2-(t_{i+1}-t_i)]^2+0\\
&\le\frac{1}{4}[(D)\sum_{i=1}^{n}\mathbf{E}[2(B_{t_{i+1}}-B_{t_i})^4+2(t_{i+1}-t_i)^2]\\
&=\frac{1}{2}[(D)\sum_{i=1}^{n}\mathbf{E}[(B_{t_{i+1}}-B_{t_i})^4]+\mathbf{E}[(t_{i+1}-t_i)^2]]\\
&=\frac{1}{2}[(D)\sum_{i=1}^{n}3(t_{i+1}-t_i)^2+(t_{i+1}-t_i)^2]\\
&=2(D)\sum_{i=1}^{n}(t_{i+1}-t_i)^2\\
&\le 2\delta(D)\sum_{i=1}^{n}(t_{i+1}-t_i)\\
&=2\delta T=2(\frac{\epsilon}{2(T+1)})T\le\epsilon.
\end{aligned}
$$

Thus

$$(GI)\int_0^T B_t\,dB_t=\frac{1}{2}B_T^2-\frac{1}{2}T.$$

□

Example 3.1.5. *Let $f : \Omega \times [a, b] \to \mathbb{R}$ be an $L^2(\Omega)$-process such that $\mathbf{E}(f_t)^2 = 0$ for all $t \in [a, b]$ except on a set of Lebesgue measure zero. Then f is GI-integrable on $[a, b]$ and*

$$(GI) \int_a^b f_t \, dB_t = 0.$$

Proof. Let $E_k = \{\xi \in [a, b] : k - 1 \leq \mathbf{E}(f_\xi)^2 < k\}$, for $k \in \mathbb{N}$. Then each E_k is a set of Lebesgue measure zero. Let $\epsilon > 0$. For each k, there exists a countable collection $\{I_{k_j}\}_{j=1}^{\infty}$ of open intervals such that $E_k \subseteq \bigcup_{j=1}^{\infty} I_{k_j}$ and $\sum_{j=1}^{\infty} |I_{k_j}| < \frac{\epsilon}{k2^k}$. Suppose $\xi \in E_k$, we choose a positive number $\delta(\xi)$ such that wherever $[u, v] \subseteq (\xi - \delta(\xi), \xi + \delta(\xi))$ we have $[u, v] \subseteq \bigcup_{j=1}^{\infty} I_{k_j}$. If $\xi \notin \bigcup_{k=1}^{\infty} E_k$, the value of $\delta(\xi)$ can be arbitrary. Let $D = \{([t_i, t_{i+1}], \xi_i)\}_{i=1}^{n}$ be a δ-fine McShane division of $[a, b]$ and $D_k \subseteq D$ such that each tag in D_k belongs to E_k.

Then, by Lemma 1.4.3 (vii),

$$(D)\sum_{i<j} \mathbf{E}[(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})(B_{t_{i+1}} - B_{t_i}))(\mathbf{E}(f_{\xi_j}|\mathcal{F}_{t_j})(B_{t_{j+1}} - B_{t_j}))] = 0.$$

Hence, we can see that

$$\begin{aligned}
\mathbf{E}(S(f,\delta,D))^2 =& \mathbf{E}[(D)\sum_{i=1}^{n}\mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})(B_{t_{i+1}}-B_{t_i})]^2 \\
=& \mathbf{E}[(D)\sum_{i=1}^{n}(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})(B_{t_{i+1}}-B_{t_i}))^2 \\
& + (D)\sum_{i<j}(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})(B_{t_{i+1}}-B_{t_i}))(\mathbf{E}(f_{\xi_j}|\mathcal{F}_{t_j})(B_{t_{j+1}}-B_{t_j}))] \\
=& (D)\sum_{i=1}^{n}\mathbf{E}[(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})(B_{t_i}-B_{t_{i+1}}))^2]+0 \\
=& (D)\sum_{i=1}^{n}\mathbf{E}[\mathbf{E}((\mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i}))^2(B_{t_i}-B_{t_{i+1}})^2|\mathcal{F}_{t_i})] \\
=& (D)\sum_{i=1}^{n}\mathbf{E}[(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i}))^2\mathbf{E}((B_{t_i}-B_{t_{i+1}})^2|\mathcal{F}_{t_i})] \\
=& (D)\sum_{i=1}^{n}\mathbf{E}[\mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})^2](t_{i+1}-t_i) \\
=& \sum_{k=1}^{\infty}(D_k)\sum_{i}\mathbf{E}[\mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})^2](t_{i+1}-t_i) \\
\leq& \sum_{k=1}^{\infty}(D_k)\sum_{i}\mathbf{E}[\mathbf{E}(f_{\xi_i}^2|\mathcal{F}_{t_i})](t_{i+1}-t_i) \\
=& \sum_{k=1}^{\infty}(D_k)\sum_{i}\mathbf{E}(f_{\xi_i}^2)(t_{i+1}-t_i) \\
\leq& \sum_{k=1}^{\infty}k(D_k)\sum_{i}(t_{i+1}-t_i) \\
\leq& \sum_{k=1}^{\infty}k\frac{\epsilon}{k2^k} \\
=& \epsilon.
\end{aligned}$$

Therefore, f is GI-integrable on $[a,b]$ and $(GI)\int_a^b f_t\ dB_t = 0$ $\square$

Theorem 3.1.6. *Let $f:\Omega\times[a,b]\to\mathbb{R}$ be an $L^2(\Omega)$-process. Then f is GI-integrable on $[a,b]$ if and only if there exist a function $F\in L^2(\Omega)$, a decreasing* *sequence* $\{\delta_n(\xi)\}_{n\in\mathbb{N}}$ *of positive functions defined on* $[a,b]$ *such that we have*

$$\lim_{n\to\infty} \mathbf{E}[S(f,\delta_n,D_n) - F]^2 = 0$$

for any δ_n*-fine McShane divisions* D_n*,* $n = 1, 2, \ldots.$

Proof. The proof is standard. Let $\epsilon > 0$ be given. Assume that f is GI-integrable to $F \in L^2(\Omega)$ on $[a,b]$, then there exists a positive function δ on $[a,b]$ such that for every δ-fine McShane division D of $[a,b]$ we have

$$\mathbf{E}[S(f,\delta,D) - F]^2 \leq \epsilon.$$

Then there exists a decreasing sequence $\{\delta_n(\xi)\}_{n\in\mathbb{N}}$ of positive functions such that for any δ_n-fine McShane division D on $[a,b]$, we have

$$\mathbf{E}[S(f,\delta_n,D_n) - F]^2 \leq \frac{1}{n}.$$

Thus, we can conclude that

$$\lim_{n\to\infty} \mathbf{E}[S(f,\delta_n,D_n) - F]^2 = 0.$$

Conversely, assume that there exists a function $F \in L^2(\Omega)$ and a decreasing sequence $\{\delta_n(\xi)\}_{n\in\mathbb{N}}$ of positive functions on $[a,b]$ such that

$$\lim_{n\to\infty} \mathbf{E}[S(f,\delta_n,D_n) - F]^2 = 0.$$

Suppose that f is not GI-integrable on $[a,b]$. Then there exists $\epsilon > 0$ such that for every positive function δ on $[a,b]$, there exists a δ-fine McShane division D of $[a,b]$ with

$$\mathbf{E}[S(f,\delta,D) - F]^2 \geq \epsilon.$$

Hence for each δ_n, there exists a δ_n-fine division D_n of $[a,b]$ such that

$$\mathbf{E}[S(f,\delta_n,D_n) - F]^2 \geq \epsilon.$$

It contradicts the fact that $\lim_{n\to\infty} \mathbf{E}[S(f,\delta_n,D_n) - F]^2 = 0$. Then we can conclude that f is GI-integrable on $[a,b]$. $\square$

3.2 Basic properties

In this section, we shall prove some basic properties of the generalized Itô integral and establish the Cauchy Criterion for the generalized Itô integral. Their proofs are standard and similar to the corresponding proofs in Section 2.2.

Theorem 3.2.1. *Let $\alpha \in \mathbb{R}$. If $f, g : \Omega \times [a, b] \to \mathbb{R}$ are GI-integrable on $[a, b]$, then*

(i) $f + g$ is GI-integrable on $[a, b]$, and

$$(GI)\int_a^b (f+g)\ dB = (GI)\int_a^b f\ dB + (GI)\int_a^b g\ dB.$$

(ii) αf is GI-integrable on $[a, b]$, and

$$(GI)\int_a^b (\alpha f)\ dB = \alpha \cdot (GI)\int_a^b f\ dB.$$

Proof. The proof is standard. (i) Let $\epsilon > 0$ and $\alpha \in \mathbb{R}$. Assume that f and g are GI-integrable on $[a, b]$ such that

$$(GI)\int_a^b f\ dB = F,$$

and

$$(GI)\int_a^b g\ dB = G.$$

Then there exist positive functions δ_1 and δ_2 on $[a, b]$ such that

$$\mathbf{E}[S(f, \delta_1, D_1) - F]^2 \leq \frac{\epsilon}{4},$$

and

$$\mathbf{E}[S(g, \delta_2, D_2) - G]^2 \leq \frac{\epsilon}{4}$$

for every δ_1, δ_2-fine McShane divisions D_1, D_2 of $[a, b]$ respectively.

Pick $\delta(\xi) = \min\{\delta_1(\xi), \delta_2(\xi)\}$. Then, for every δ-fine McShane division D of $[a, b]$ we have, $\mathbf{E}[S(f, \delta, D) - F]^2 \leq \frac{\epsilon}{4}$ and $\mathbf{E}[S(g, \delta, D) - G]^2 \leq \frac{\epsilon}{4}$.

Since

$$\begin{aligned} S(f+g,\delta,D) =&(D)\sum_{i=1}^{n}\mathbf{E}(f_{\xi_i}+g_{\xi_i}|\mathcal{F}_{t_i})(B_{t_{i+1}}-B_{t_i}) \\ =&(D)\sum_{i=1}^{n}\mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})(B_{t_{i+1}}-B_{t_i}) \\ &+(D)\sum_{i=1}^{n}\mathbf{E}(g_{\xi_i}|\mathcal{F}_{t_i})(B_{t_{i+1}}-B_{t_i}) \\ =&S(f,\delta,D)+S(g,\delta,D). \end{aligned}$$

Then,

$$\begin{aligned} \mathbf{E}[S(f+g,\delta,D)-(F+G)]^2 =&\mathbf{E}[(S(f,\delta,D)-F)+(S(g,\delta,D)-G)]^2 \\ =&\mathbf{E}[(S(f,\delta,D)-F)^2+(S(g,\delta,D)-G)^2) \\ &+2(S(f,\delta,D)-F)(S(g,\delta,D)-G)] \\ \leq&\mathbf{E}[2(S(f,\delta,D)-F)^2+2(S(g,\delta,D)-G)^2] \\ =&2\mathbf{E}[S(f,\delta,D)-F]^2+2\mathbf{E}[S(g,\delta,D)-G]^2 \\ \leq&2(\frac{\epsilon}{4})+2(\frac{\epsilon}{4})=\epsilon. \end{aligned}$$

Hence, $f+g$ is GI-integrable on $[a,b]$ and

$$\begin{aligned} (GI)\int_a^b (f+g)\ dB =&F+G \\ =&(GI)\int_a^b f\ dB+(GI)\int_a^b g\ dB. \end{aligned}$$

(ii) It is obvious that

$$(GI)\int_a^b 0\cdot f\ dB=0=0\cdot(GI)\int_a^b f\ dB.$$

Now, we assume that $\alpha\neq 0$. There exists a positive function δ on $[a,b]$ such that for every δ-fine division D of $[a,b]$

$$\mathbf{E}[S(f,\delta,D)-F]^2\leq\frac{\epsilon}{\alpha^2}.$$

Since

$$\begin{aligned}S(\alpha f,\delta,D) =&(D)\sum_{i=1}^{n}\mathbf{E}(\alpha f_{\xi_i}|\mathcal{F}_{t_i})(B_{t_{i+1}}-B_{t_i})\\ =&\alpha(D)\sum_{i=1}^{n}\mathbf{E}(f_{\xi_i}|\mathcal{F}_{t_i})(B_{t_{i+1}}-B_{t_i})\\ =&\alpha S(f,\delta,D).\end{aligned}$$

Then

$$\begin{aligned}\mathbf{E}[S(\alpha f,\delta,D)-\alpha F]^2 =&\mathbf{E}[\alpha S(f,\delta,D)-\alpha F]^2\\ =&\alpha^2\mathbf{E}[S(f,\delta,D)-F]^2\\ \leq&\alpha^2(\frac{\epsilon}{\alpha^2})=\epsilon.\end{aligned}$$

Hence αf is GI-integrable on $[a,b]$ and

$$(GI)\int_a^b \alpha f\ dB=\alpha F=\alpha\cdot(GI)\int_a^b f\ dB.$$

$\square$

Definition 3.2.2. Let $f:\Omega\times[a,b]\to\mathbb{R}$ be an $L^2(\Omega)$-process, $A\subseteq[a,b]$ and $\mathcal{X}_A$ be the characteristic function of $\Omega\times A$. Then f is said to be GI-integrable on A if $f\cdot\mathcal{X}_A$ is GI-integrable on A.

$$(GI)\int_a^b f\cdot\mathcal{X}_A\ dB \text{ is denoted by } (GI)\int_A f\ dB.$$

If $A=[c,d]$, then $(GI)\int_A f\ dB$ is denoted by $(GI)\int_c^d f\ dB$.

Theorem 3.2.3. *If $f:\Omega\times[a,b]\to\mathbb{R}$ is a GI-integrable on [a,c] and [c,b], then f is GI-integrable on $[a,b]$ and*

$$(GI)\int_a^b f\ dB=(GI)\int_a^c f\ dB+(GI)\int_c^b f\ dB.$$

Proof. By Definition 3.2.2, we have

$$(GI)\int_a^c f\ dB=(GI)\int_a^b f\cdot\mathcal{X}_{[a,c]}\ dB,$$

and
$$(GI)\int_c^b f\ dB = (GI)\int_a^b f\cdot\mathcal{X}_{[c,b]}\ dB.$$
Then, from Theorem 3.2.1, we can see that
$$\begin{aligned}(GI)\int_a^b f\ dB =&(GI)\int_a^b (f\cdot\mathcal{X}_{[a,c]} + f\cdot\mathcal{X}_{(c,b]})\ dB\\ =&(GI)\int_a^b f\cdot\mathcal{X}_{[a,c]}\ dB + (GI)\int_a^b f\cdot\mathcal{X}_{(c,b]}\ dB\\ =&(GI)\int_a^c f\ dB + (GI)\int_{(c,b]} f\ dB.\end{aligned}$$
By Example 3.1.3 (ii), $(GI)\int_{(c,b]} f\ dB = (GI)\int_c^b f\ dB$. Then we have the required result. $\square$

Theorem 3.2.4 (Cauchy Criterion for generalized Itô integrals)**.** *Let $f : \Omega\times[a,b]\to\mathbb{R}$ be an $L^2(\Omega)$-process. Then f is GI-integrable on $[a,b]$ if and only if for every $\epsilon > 0$, there exists a positive function δ on $[a,b]$ such that for any two δ-fine McShane divisions of $[a,b]$, $D = \{([t_i, t_{i+1}], \xi_i)\}$ and $D' = \{([t'_j, t'_{j+1}], \xi'_j)\}$, we have*
$$\mathbf{E}[S(f,\delta,D) - S(f,\delta,D')]^2 \leq \epsilon.$$

Proof. The proof is standard. Assume that f is GI-integrable with respect to B on $[a,b]$.

Let $\epsilon > 0$ be given. Then there exists a positive function δ on $[a,b]$ such that for any two δ-fine divisions D and D' of $[a,b]$, we have
$$\mathbf{E}[S(f,\delta,D) - F]^2 \leq \frac{\epsilon}{4},$$
and
$$\mathbf{E}[S(f,\delta,D') - F]^2 \leq \frac{\epsilon}{4}.$$
We can see that
$$\begin{aligned}\mathbf{E}[S(f,\delta,D) - S(f,\delta,D')]^2 =&\mathbf{E}[(S(f,\delta,D) - F) - (S(f,\delta,D') - F)]^2\\ \leq&2\mathbf{E}[S(f,\delta,D) - F]^2 + 2\mathbf{E}[S(f,\delta,D') - F]^2\\ \leq&2(\frac{\epsilon}{4}) + 2(\frac{\epsilon}{4}) = \epsilon.\end{aligned}$$

Conversely, assume that for every $\epsilon > 0$ there exists a function $\delta : [a,b] \rightarrow \mathbb{R}^+$ such that for any two δ-fine McShane divisions of $[a,b]$, $D = \{[t_i, t_{i+1}], \xi_i\}$ and $D' = \{[t'_j, t'_{j+1}], \xi'_j\}$, we have

$$\mathbf{E}[S(f,\delta,D) - S(f,\delta,D')]^2 \leq \frac{\epsilon}{4},$$

Let $\epsilon_n = \frac{1}{n}$, for $n \in \mathbb{N}$ and δ_n be the corresponding positive function on $[a,b]$. We may assume that if $m \geq n$, $\delta_m(\xi) \leq \delta_n(\xi)$ for each $\xi \in [a,b]$. Then a δ_n-fine division of $[a,b]$ is also a δ_n-fine McShane division of $[a,b]$. Hence

$$\mathbf{E}(|S(f,\delta_{m_1},D_{m_1}) - S(f,\delta_{m_2},D_{m_2})|)^2 \leq \frac{1}{n}$$

whenever $m_1, m_2 \geq n$.

We conclude that $\{S(f,\delta_n,D_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$-space. Therefore, there exists $F \in L^2(\Omega)$ such that $\lim\limits_{n\to\infty} S(f,\delta_n,D_n) = F$ under L^2-norm.

Let $\epsilon > 0$ be given. Then there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$,

$$\mathbf{E}(|S(f,\delta_n,D_n) - F|)^2 \leq \frac{\epsilon}{4}.$$

Let $\frac{1}{N_2} \leq \frac{\epsilon}{4}$ and $N = max\{N_1, N_2\}$. Let $\delta(\xi) = \delta_N(\xi)$ for each $\xi \in [a,b]$. Then

$$\begin{aligned}\mathbf{E}(|S(f,\delta,D) - F|)^2 =& \mathbf{E}(|(S(f,\delta,D) - S(f,\delta_N,D_N)) + (S(f,\delta_N,D_N) - F)|)^2 \\ \leq& 2[\mathbf{E}(|S(f,\delta,D) - S(f,\delta_N,D_N)|)]^2 + 2[\mathbf{E}(|S(f,\delta_N,D_N) - F|)]^2 \\ \leq& 2(\frac{1}{N}) + 2(\frac{\epsilon}{4}) = \epsilon.\end{aligned}$$

Thus f is GI-integrable on $[a,b]$. Moreover, we also know that $(GI)\displaystyle\int_a^b f\ dt = F$. $\square$

Theorem 3.2.5. *If f is GI-integrable on $[a,b]$, then f is also GI-integrable on every subinterval $[c,d]$ of $[a,b]$.*

Proof. The proof is similar to that of Theorem 2.2.6. We shall only prove the case that f is GI-integrable on $[a, c]$. Let f be GH-integrable on $[a, b]$. Let $\epsilon > 0$ be given. By Cauchy Criterion for generalized Itô integrals, there exists a positive function δ on $[a, b]$ such that for any two δ-fine McShane divisions D, D' of $[a, b]$, we have

$$\mathbf{E}(|S(f, \delta, D) - S(f, \delta, D')|) \leq \epsilon.$$

Choose

$$\delta'(\xi) = \begin{cases} \min\{\delta(\xi), c - \xi, \epsilon\} & , \text{ if } \xi \in [a, c); \\ \min\{\delta(\xi), \epsilon\} & , \text{ if } \xi = c; \\ \min\{\delta(\xi), \xi - c, \epsilon\} & , \text{ if } \xi \in (c, b]. \end{cases}$$

It is clear that $\delta'(\xi) \leq \delta(\xi)$. Hence, every δ'-fine McShane division D is also δ-fine.

Let $D_1 = \{([t_i, t_{i+1}], \xi_i)\}_{i=1}^m$ and $D_2 = \{([t'_j, t'_{j+1}], \xi'_j)\}_{j=1}^n$ be δ'-fine McShane divisions of $[a, b]$.

It is clear that c always is a tag, we may assume that $\xi_g = c = \xi'_h$. Let D'_1 be a partial division of D_1 by deleting $(c, t_{g+1}]$ and $\{[t_i, t_{i+1}]\}_{i=g+1}^m$ from D_1, we do the same thing for D_2 and obtain D'_2. Thus D'_1 and D'_2 form δ-fine McShane divisions of $[a, c]$

Let D_3 be a δ-fine McShane division of $[c, b]$. Hence $D'_1 \cup D_3$ and $D'_2 \cup D_3$ form δ-fine full McShane divisions of $[a, b]$. Then,

$$\begin{aligned}
&\mathbf{E}(S(f\mathcal{X}_{[a,c]}, \delta', D_1) - S(f\mathcal{X}_{[a,c]}, \delta', D_2))^2 \\
=&\mathbf{E}(S(f, \delta', D'_1) + \mathbf{E}(f_c|\mathcal{F}_{t_g})(B_{t_{g+1}} - B_c) - S(f, \delta', D'_1) - \mathbf{E}(f_c|\mathcal{F}_{t'_h})(B_{t'_{h+1}} - B_c))^2 \\
\leq&3\mathbf{E}([\mathbf{E}(f_c|\mathcal{F}_{t_g})]^2(t_{g+1} - c) + [\mathbf{E}(f_c|\mathcal{F}_{t'_h})]^2(t'_{h+1} - c)) \text{ (for more detail see Lemma 3.3.2)} \\
&\quad + 3\mathbf{E}(S(f, \delta, D'_1 \cup D_3) - S(f, \delta, D'_2 \cup D_3))^2 \\
\leq&6\mathbf{E}(f_c^2)\epsilon + \epsilon.
\end{aligned}$$

Hence f is GI-integrable on $[a, c]$.

Similarly, we can prove the case for $[d, b]$. Since $f\mathcal{X}_{[c,d]} = f - f\mathcal{X}_{[a,c)} - f\mathcal{X}_{(d,b]}$ and $\int_a^b f \cdot \mathcal{X}_{\{c\}}\, dB = \int_a^b f \cdot \mathcal{X}_{\{d\}}\, dB = 0$. Hence f is GI-integrable on $[c, d]$. $\square$

3.3 Stochastic properties

In this section we shall derive some stochastic properties of the generalized Itô integral. The ideas of the following proofs are based on the ideas of the corresponding proofs for the Itô integral, see [5, 22].

Lemma 3.3.1. *Let $f : \Omega \times [a, b] \to \mathbb{R}$ be GI-integrable on $[a, b]$ and $F(A) = (GI)\int_A f_t\, dB_t$, where A is a subinterval of $[a, b]$. Let $I = [c, d]$ and $J = [u, v]$ be two non-overlapping subintervals of $[a, b]$. Then*

(i) F has the orthogonal increment property, that is, $\mathbf{E}(F(I)F(J)) = 0$,

(ii) $\mathbf{E}(B(I)F(J)) = 0$,

(iii) $\mathbf{E}[\mathbf{E}((f_\xi|\mathcal{F}_c)B(I) - F(I))(\mathbf{E}(f_\eta|\mathcal{F}_u)B(J) - F(J))] = 0$, where $\xi, \eta \in [a, b]$.

Proof. Let $[a_1, b_1]$ and $[a_2, b_2]$ be subintervals of $[a, b]$ such that $b_1 \le a_2$. Then for $\xi_1, \xi_2 \in [a, b]$, by Lemma 1.4.3 (vii), we have

$$\mathbf{E}[\mathbf{E}(f_{\xi_1}|\mathcal{F}_{a_1})(B_{b_1} - B_{a_1})\mathbf{E}(f_{\xi_2}|\mathcal{F}_{a_2})(B_{b_2} - B_{a_2})] = 0.$$

Similarly, if $I = [c, d]$ and $[a_i, b_i]$ are non-overlapping subintervals of $[a, b]$, then

$$\mathbf{E}[B(I)\mathbf{E}(f_{\xi_i}|\mathcal{F}_{a_i})(B_{b_i} - B_{a_i})] = 0.$$

Hence, for all positive functions δ_n, δ_m, we have

$$\mathbf{E}[S(f, \delta_n, D(I))S(f, \delta_m, D(J))] = 0,$$

where I and J are non-overlapping subintervals of $[a, b]$, $D(I)$ and $D(J)$ refer to the divisions of I and J, respectively. From Theorem 3.1.6, pick a sequence $\{\delta_n\}_{n\in\mathbb{N}}$

of positive functions such that

$$\lim_{n\to\infty} \mathbf{E}(S(f,\delta_n,D(I)) - F(I))^2 = 0$$

and

$$\lim_{n\to\infty} \mathbf{E}(S(f,\delta_n,D(J)) - F(J))^2 = 0.$$

Then

$$\mathbf{E}[F(I)F(J)] = \lim_{n\to\infty} \mathbf{E}[S(f,\delta_n,D(I))S(f,\delta_m,D(J))] = 0.$$

and

$$\mathbf{E}[B(I)F(J)] = \lim_{n\to\infty} \mathbf{E}[B(I)S(f,\delta_n,D(J))] = 0.$$

Hence (i) and (ii) hold. Furthermore, (iii) follows directly from (i) and (ii). □

Lemma 3.3.2. *Let f be GI-integrable on $[a,b]$ and $F(u,v) = (GI)\int_u^v f_t\, dB_t$. Let $D = \{([u_i,v_i],\xi_i)\}_{i=1}^n$ be a δ-fine partial McShane division of $[a,b]$. Then*

(i) $\mathbf{E}[(D)\sum_{i=1}^{n}\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i} - B_{u_i})]^2 = (D)\sum_{i=1}^{n}\mathbf{E}[(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2](v_i - u_i)$

(ii) $\mathbf{E}[(D)\sum_{i=1}^{n}\{\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i} - B_{u_i}) - F(u_i,v_i)\}]^2$

$$= \mathbf{E}[(D)\sum_{i=1}^{n}\{\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i} - B_{u_i}) - F(u_i,v_i)\}^2].$$

Proof. Let f be GI-integrable on $[a,b]$ and $F(u,v) = (GI)\int_u^v f_t\, dB_t$. Let $D = \{([u_i,v_i],\xi_i)\}_{i=1}^n$ be a δ-fine partial McShane division of $[a,b]$. Then

(i)

$$\begin{aligned}
\mathbf{E}[(D)&\sum_{i=1}^{n}\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i} - B_{u_i})]^2 \\
&= \mathbf{E}[(D)\sum_{i=1}^{n}(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i} - B_{u_i}))^2 \\
&\quad + (D)\sum_{i\neq j}(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i} - B_{u_i})\mathbf{E}(f_{\xi_j}|\mathcal{F}_{t_j})(B_{t_{j+1}} - B_{t_j}))].
\end{aligned}$$

By Lemma 1.4.3 (vi), we have

$$\mathbf{E}[(D)\sum_{i=1}^{n}(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i}-B_{u_i}))^2] = (D)\sum_{i=1}^{n}\mathbf{E}[(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2(v_i-u_i)]$$

and, by Lemma 1.4.3 (vii), we have

$$\mathbf{E}[(D)\sum_{i\neq j}(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i}-B_{u_i})\mathbf{E}(f_{\xi_j}|\mathcal{F}_{t_j})(B_{t_{j+1}}-B_{t_j}))] = 0.$$

Hence,

$$\mathbf{E}[(D)\sum_{i=1}^{n}\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i}-B_{u_i})]^2 = (D)\sum_{i=1}^{n}\mathbf{E}[(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2](v_i-u_i).$$

(ii)

$$\begin{aligned}
&\mathbf{E}[(D)\sum_{i=1}^{n}\{\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i}-B_{u_i})-F(u_i,v_i)\}]^2 \\
=&\mathbf{E}[(D)\sum_{i=1}^{n}\{\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i}-B_{u_i})-F(u_i,v_i)\}^2 \\
&+\sum_{i\neq j}\{\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i}-B_{u_i})-F(u_i,v_i)\}\{\mathbf{E}(f_{\xi_j}|\mathcal{F}_{u_j})(B_{v_j}-B_{u_j})-F(u_j,v_j)\}] \\
=&\mathbf{E}[(D)\sum_{i=1}^{n}\{\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i}-B_{u_i})-F(u_i,v_i)\}^2+0] \text{ (by Lemma 3.3.1 (iii))} \\
=&\mathbf{E}[(D)\sum_{i=1}^{n}\{\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i}-B_{u_i})-F(u_i,v_i)\}^2].
\end{aligned}$$

□

Lemma 3.3.3 (Henstock's lemma). *Let $f:\Omega\times[a,b]\to\mathbb{R}$ be GI-integrable on $[a,b]$ and $F(u,v)=(GI)\int_u^v f_t\, dB_t$ for any $[u,v]\subseteq[a,b]$. Then for every $\epsilon>0$, there exists a positive function δ on $[a,b]$ such that for every δ-fine partial McShane division $D=\{([u_i,v_i],\xi_i)\}_{i=1}^n$ of $[a,b]$ we have*

$$\mathbf{E}[(D)\sum_{i=1}^{n}\{\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i}-B_{u_i})-F(u_i,v_i)\}^2]\le\epsilon.$$

Proof. Let $\epsilon > 0$ be given. Assume that f is GI-integrable on $[a, b]$. Then there exists a positive function δ on $[a, b]$ such that for every δ-fine McShane division $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ on $[a, b]$, we have

$$\mathbf{E}[(D)\sum_{i=1}^{n} \mathbf{E}(f_{\xi_i}|\mathcal{F}_i)(B_{v_i} - B_{u_i}) - F(u_i, v_i)]^2 \leq \epsilon.$$

Hence, we can see that

$$\begin{aligned}
&\mathbf{E}[(D)\sum_{i=1}^{n}\{\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i} - B_{u_i}) - F(u_i, v_i)\}^2] \\
&=\mathbf{E}[(D)\sum_{i=1}^{n}\{\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i} - B_{u_i}) - F(u_i, v_i)\}]^2 \text{ (from Lemma 3.3.2 (ii))} \\
&\leq\epsilon.
\end{aligned}$$

Therefore, the above inequality also holds for any δ-fine partial (not full) McShane division D of $[a, b]$. □

Theorem 3.3.4. *Let f be GI-integrable on $[a, b]$. Then*

(i) $\mathbf{E}(\int_a^b f_t \, dB_t) = 0.$

(ii) $\mathbf{E}(\sum_{i=1}^{n} \int_{u_i}^{v_i} f_t \, dB_t)^2 = \sum_{i=1}^{n} \mathbf{E}(\int_{u_i}^{v_i} f_t \, dB_t)^2$ *for any finite collections* $\{[u_i, v_i]\}_{i=1}^n$ *of non-overlapping subintervals of* $[a, b]$.

Proof. Let f be GI-integrable on $[a, b]$. Then, there exist δ_n-fine McShane divisions D_n, $n = 1, 2, ...,$ such that

$$\lim_{n\to\infty} \mathbf{E}[(D_n)\sum_{i=1}^{p(n)} \mathbf{E}(f_{\xi_i^{(n)}}|\mathcal{F}_{t_i^{(n)}})(B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}}) - \int_a^b f_t \, dB_t]^2 = 0,$$

Hence

$$\lim_{n\to\infty} \mathbf{E}[(D_n)\sum_{i=1}^{p(n)} \mathbf{E}(f_{\xi_i^{(n)}}|\mathcal{F}_{t_i^{(n)}})(B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}}) - \int_a^b f_t \, dB_t] = 0.$$

(i) It is easy to see that

$$\begin{aligned}
\mathbf{E}(\int_a^b f_t \ dB_t) &= \lim_{n\to\infty}(D_n)\mathbf{E}(\sum_{i=1}^n \mathbf{E}(f_{\xi_i^{(n)}}|\mathcal{F}_{t_i^{(n)}})(B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}}))\\
&= \lim_{n\to\infty}(D_n)\sum_{i=1}^n \mathbf{E}[\mathbf{E}(f_{\xi_i^{(n)}}|\mathcal{F}_{t_i^{(n)}})(B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}})]\\
&= \lim_{n\to\infty}(D_n)\sum_{i=1}^n \mathbf{E}[\mathbf{E}(\mathbf{E}(f_{\xi_i^{(n)}}|\mathcal{F}_{t_i^{(n)}})(B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}})|\mathcal{F}_{t_i^{(n)}})]\\
&= \lim_{n\to\infty}(D_n)\sum_{i=1}^n \mathbf{E}[\mathbf{E}(f_{\xi_i^{(n)}}|\mathcal{F}_{t_i^{(n)}})\mathbf{E}((B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}})|\mathcal{F}_{t_i^{(n)}})]\\
&= \lim_{n\to\infty}(D_n)\sum_{i=1}^n \mathbf{E}[\mathbf{E}(f_{\xi_i^{(n)}}|\mathcal{F}_{t_i^{(n)}})(B_{t_i^{(n)}} - B_{t_i^{(n)}})]\\
&=0.
\end{aligned}$$

Next, let $\{[u_i, v_i]\}_{i=1^n}$ be any finite collection of non-overlapping subintervals of $[a, b]$. Then,

$$\mathbf{E}(\sum_{i\neq j}(\int_{u_i}^{v_i} f_t \ dB_t)(\int_{u_j}^{v_j} f_t \ dB_t)) = 0 \text{ (by Lemma 3.3.1 (i)).}$$

Therefore, we can see that

$$\begin{aligned}
\mathbf{E}(\sum_{i=1}^n \int_{u_i}^{v_i} f_t \ dB_t)^2 =&\mathbf{E}(\sum_{i=1}^n(\int_{u_i}^{v_i} f_t \ dB_t)^2 + \sum_{i\neq j}(\int_{u_i}^{v_i} f_t \ dB_t)(\int_{u_j}^{v_j} f_t \ dB_t))\\
=&\mathbf{E}(\sum_{i=1}^n(\int_{u_i}^{v_i} f_t \ dB_t)^2)\\
=&\sum_{i=1}^n \mathbf{E}(\int_{u_i}^{v_i} f_t \ dB_t)^2.
\end{aligned}$$

$\square$

3.4 Absolute continuity

In this section, as in Section 2.4, we shall prove the absolute continuity of the GI-integral.

Definition 3.4.1. A stochastic process $\mathbf{X} : \Omega \times [a,b] \to \mathbb{R}$ is said to be $AC^2([a,b])$ if for every $\epsilon > 0$, there exists a positive real number η such that for any partial partition $D = \{[u_i, v_i]\}_{i=1}^n$ of $[a,b]$, with $(D)\sum_{i=1}^{n}(v_i - u_i) \leq \eta$, we have

$$\mathbf{E}[(D)\sum_{i=1}^{n}(\mathbf{X}_{v_i} - \mathbf{X}_{u_i})^2] \leq \epsilon.$$

Theorem 3.4.2. *If f is GI-integrable to F on $[a,b]$, then F has the $AC^2([a,b])$ property.*

Proof. The proof is similar to that of Theorem 2.4.2. Let f be GI-integrable to F and $\epsilon > 0$ be given. Then by Lemma 3.3.3 (Henstock's Lemma), there exists a positive function δ on $[a,b]$ such that for every δ-fine partial McShane division $D = \{([u_i, v_i], \xi_i)\}$, we have

$$\mathbf{E}((D)\sum |(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))(B_{v_i} - B_{u_i}) - F(u_i, v_i)|^2) \leq \frac{\epsilon}{2}.$$

Hence,

$$\begin{aligned} |(D)\sum \mathbf{E}(F(u_i, v_i))^2| &\leq |(D)\sum \mathbf{E}((\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2(v_i - u_i))| + \frac{\epsilon}{2} \\ &\leq (D)\sum \mathbf{E}(f_{\xi_i}^2)(v_i - u_i) + \frac{\epsilon}{2}. \end{aligned}$$

Let $D' = \{([t_i, t_{i+1}], \xi_i)\}_{i=1}^n$ be a fixed δ-fine McShane division of $[a,b]$ and $M > 0$ with

$$M \geq \max_{1 \leq i \leq n}\{\mathbf{E}(f_{\xi_i}^2)\}.$$

Choose $\eta \leq \frac{\epsilon}{2M}$. Let $\{[x_i, y_i]\}$ be a partial partition of $[a,b]$. Assume that $\sum |y_j - x_j| \leq \eta$. Let the refinement of $\{[x_i, y_i]\}$ and $\{[t_i, t_{i+1}]\}$ on $\cup_i [x_i, y_i]$ be $\{[a_k, b_k]\}_{k=1}^q$.

Then $\cup_k[a_k, b_k] = \cup_i[x_i, y_i]$. If $[a_k, b_k]$ is a subinterval of $[t_i, t_{i+1}]$, then we choose ξ_i as an associate point of $[a_k, b_k]$, denote ξ_i by η_k. From this construction we get new δ-fine partial McShane division $D'' = \{([a_k, b_k], \eta_k)\}_{k=1}^q$. Hence, we can see that

$$\begin{aligned}
|\sum \mathbf{E}(F(x_j, y_j))^2| =&|(D'')\sum_{k=1}^{q} \mathbf{E}(F(a_k, b_k))|^2 \\
\leq&(D'')\sum_{k=1}^{q} \mathbf{E}(f_{\eta_k}^2)(b_k - a_k) + \frac{\epsilon}{2} \\
\leq&M(D'')\sum_{k=1}^{q}(b_k - a_k) + \frac{\epsilon}{2} \\
\leq&M\frac{\epsilon}{2M} + \frac{\epsilon}{2} \\
=&\epsilon.
\end{aligned}$$

Therefore, F is an $AC^2([a, b])$ process. □

Corollary 3.4.3. *If f is a GI-integrable process on $[a, b]$, then for every $\epsilon > 0$, there exist a positive function δ and a positive real number η such that for any δ-fine partial McShane division $D = \{([u_i, v_i], \xi_i)\}$ of $[a, b]$ with $(D)\sum(v_i - u_i) \leq \eta$, we have*

$$\mathbf{E}[(D)\sum(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2(v_i - u_i)] \leq \epsilon$$

Proof. The proof is similar to that of Corollary 2.4.3. Let f be a GI-integrable process and $\epsilon > 0$ be given. From Theorem 3.4.2, there exists a positive real number η such that for any partial partition $D = \{[u_i, v_i]\}$ of $[a, b]$ with $(D)\sum|v_i - u_i| \leq \eta$, we have

$$|\sum_{i=1}^{n} \mathbf{E}(F(u_i, v_i))^2| \leq \frac{\epsilon}{2}.$$

From Henstock's Lemma, there exists a positive function δ on $[a, b]$ such that for any δ-fine partial McShane partition $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ we have

$$\mathbf{E}((D)\sum_{i=1}^{n} |(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))(B_{v_i} - B_{u_i}) - F(u_i, v_i)|^2) \leq \frac{\epsilon}{2}.$$

Hence, we can see that

$$\begin{aligned}\mathbf{E}[\sum_{i=1}^{n}(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2(v_i-u_i)] =&\mathbf{E}[\sum_{i=1}^{n}|(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}))^2(B_{v_i}-B_{u_i})-F(u_i,v_i)|^2]\\ &+\mathbf{E}[\sum_{i=1}^{n}(F(u_i,v_i))]\\ \leq&\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

□

3.5 Integrable processes

In this section, we shall prove that if f is an adapted $L^2(\Omega)$-process, then f is GI-integrable if and only if f^2 is M-integrable (Bochner integrable) on $[a,b]$. This is our main result of this chapter.

Lemma 3.5.1. *If f^2 is M-integrable (Bochner integrable) on $[a,b]$, then f is GI-integrable on $[a,b]$.*

Proof. Suppose f^2 is M-integrable on $[a,b]$. Then $|f|^2$ is M-integrable on $[a,b]$. Observe that $f^+=\{f_t^+ : t\in[a,b]\}=\frac{|f|+f}{2}$ and $f^-=f-f^+$. Thus $(f^+)^2$ and $(f^-)^2$ are M-integrable on $[a,b]$. Therefore, in the following proof, we may assume that f is nonnegative. For each $\epsilon>0$, there exists a positive function δ on $[a,b]$ such that whenever $D_1=\{([u_i,v_i],\xi_i)\}_{i=1}^n$ and $D_2=\{([u_i,v_i],\eta_i)\}_{i=1}^n$ are two McShane δ-fine divisions of $[a,b]$, we have

$$\mathbf{E}\big(\sum_{i=1}^{n}|f_{\xi_i}^2-f_{\eta_i}^2|(v_i-u_i)\big)\leq\epsilon.$$

Observe that

$$\begin{aligned}\mathbf{E}\big(|\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})-\mathbf{E}(f_{\eta_i}|\mathcal{F}_{u_i})|^2\big) \leq&\mathbf{E}\big(\mathbf{E}(|f_{\xi_i}-f_{\eta_i}|^2|\mathcal{F}_{u_i})\big)\\ \leq&\mathbf{E}(|f_{\xi_i}-f_{\eta_i}|^2)\end{aligned}$$

and

$$(a-b)^2 \le |a^2-b^2| \text{ if } a,b \ge 0.$$

Hence

$$\begin{aligned}
&\mathbf{E}\big(\sum_{i=1}^{n}(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})-\mathbf{E}(f_{\eta_i}|\mathcal{F}_{u_i}))^2(B_{v_i}-B_{u_i})^2\big)\\
&\qquad=\mathbf{E}\big(\sum_{i=1}^{n}(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})-\mathbf{E}(f_{\eta_i}|\mathcal{F}_{u_i}))^2(v_i-u_i)\big)\\
&\qquad\le\mathbf{E}\big(\sum_{i=1}^{n}|f_{\xi_i}-f_{\eta_i}|^2(v_i-u_i)\big)\\
&\qquad\le\mathbf{E}\big(\sum_{i=1}^{n}|f_{\xi_i}^2-f_{\eta_i}^2|(v_i-u_i)\big).
\end{aligned}$$

Therefore, if f^2 is M-integrable on $[a,b]$, then, by Theorem 3.2.4, f is GI-integrable on $[a,b]$. □

Theorem 3.5.2. *[5, 22] Let $f:\Omega\times[a,b]\to\mathbb{R}$ be an adapted $L^2(\Omega)$-process. Then f is Itô -integrable on $[a,b]$ to $A\in L^2(\Omega)$ if for each $\epsilon>0$, there exist $\eta>0$ and a positive function δ on $[a,b]$ such that whenever $D=\{([u_i,v_i],\xi_i)\}_{i=1}^n$ is a δ-fine partial belated division of $[a,b]$ (see Definition 2.5.4) with $\big|b-a-\sum_{i=1}^{n}|v_i-u_i|\big|\le\eta$, we have*

$$\mathbf{E}\big(|\sum_{i=1}^{n}f_{\xi_i}(B_{v_i}-B_{u_i})-A|^2\big)\le\epsilon.$$

Theorem 3.5.3. *[5] Let $f:\Omega\times[a,b]\to\mathbb{R}$ be an adapted $L^2(\Omega)$-process. Then f is Itô -integrable on $[a,b]$ if and only if f^2 is M-integrable (Bochner integrable) on $[a,b]$.*

Lemma 3.5.4. *Let $f:\Omega\times[a,b]\to\mathbb{R}$ be an adapted $L^2(\Omega)$-process. If f is GI-integrable on $[a,b]$, then f is Itô -integrable on $[a,b]$. Their integrals are equal.*

Proof. Suppose f is GI-integrable on $[a,b]$. For each $\epsilon>0$, there exists a positive function δ on $[a,b]$ such that whenever $D=\{([u_i,v_i],\xi_i)\}_{i=1}^n$ is a δ-fine McShane division of $[a, b]$, we have

$$\mathbf{E}\bigg(\Big|\sum_{i=1}^{n}\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i}-B_{u_i})-(GI)\int_a^b f\Big|\bigg)\leq\epsilon.$$

On the other hand, from Corollary 3.4.3, there exists $\eta > 0$ such that whenever $D' = \{([x_i, y_i])\}_{i=1}^m$ is a δ-fine partial McShane division of $[a, b]$ with $\sum_{i=1}^{m}|x_i - y_i| \leq \eta$, we have

$$\mathbf{E}\bigg(\Big|\sum_{i=1}^{m}\big(\mathbf{E}(f_{\xi_i}|\mathcal{F}_{x_i})\big)^2(y_i-x_i)\Big|\bigg)\leq\epsilon.$$

Let $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ be a δ-fine partial belated division of $[a, b]$ and $D' = \{([x_i, y_i], \eta_i)\}_{i=1}^m$ be a δ-fine partial McShane division of the closure of $[a, b] \setminus \cup_{i=1}^n [u_i, v_i]$ such that $\sum_{i=1}^{m}|x_i - y_i| \leq \eta$. Then

$$\mathbf{E}\bigg(\Big|(D)\sum_{i=1}^{n}\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i})(B_{v_i}-B_{u_i})-(GI)\int_a^b f\Big|\bigg)^2\leq 2\epsilon.$$

Observe that f is adapted, so f_{ξ_i} is $\mathcal{F}_{\xi_i}$-measurable. Then f_{ξ_i} is $\mathcal{F}_{u_i}$-measurable. Consequently, $\mathbf{E}(f_{\xi_i}|\mathcal{F}_{u_i}) = f_{\xi_i}$. Therefore

$$\mathbf{E}|(D)\sum_{i=1}^{n}f_{\xi_i}(B_{v_i}-B_{u_i})-(GI)\int_a^b f|^2\leq 2\epsilon.$$

Hence, by Theorem 3.5.3, f is Itô -integrable on $[a, b]$. □

Lemma 3.5.5. *Let $f : \Omega \times [a, b] \to \mathbb{R}$ be an adapted $L^2(\Omega)$-process. If f is GI-integrable on $[a, b]$, then f^2 is M-integrable (Bochner integrable) on $[a, b]$.*

Proof. It follows from Theorem 3.5.3 and Lemma 3.5.4. □

Theorem 3.5.6. *Let $f : \Omega \times [a, b] \to \mathbb{R}$ be an adapted $L^2(\Omega)$-process. Then f is GI-integrable on $[a, b]$ if and only if f^2 is M-integrable (Bochner integrable) on $[a, b]$.*

Proof. It follows from Lemmas 3.5.1 and 3.5.5. □

3.6 Isometry

In this section, we shall prove the isometry property for the GI-integral.

Theorem 3.6.1. *[5, 22] Let f be Itô integrable on $[a, b]$. Then $\mathbf{E}(f_t^2)$ is Lebesgue integrable on $[a, b]$ and*

$$\mathbf{E}((It\hat{o})\int_a^b f\ dB)^2 = (L)\int_a^b \mathbf{E}(f_t^2)\ dt.$$

Theorem 3.6.2. *Let f be GI-integrable on $[a, b]$. Suppose f is adapted. Then $\mathbf{E}(f_t^2)$ is Lebesgue integrable on $[a, b]$ and*

$$\mathbf{E}((GI)\int_a^b f\ dB)^2 = (L)\int_a^b \mathbf{E}(f_t^2)\ dt.$$

Proof. It follows from Lemma 3.5.4 and Theorem 3.6.1. □

3.7 Dominated Convergence Theorem

Now, we can easily derive the Dominated Convergence Theorem for the GI-integral.

Theorem 3.7.1 (Dominated Convergence Theorem)**.** *Let $f^{(n)}, n = 1, 2, ...,$be a sequence of adapted GI-integrable processes defined on $[a, b]$ and f an adapted process on $[a, b]$. Suppose that*

(i) $\mathbf{E}(f_t^{(n)} - f_t)^2 \to 0$ *as* $n \to \infty$ *for almost all* $t \in [a, b]$,

(ii) $|f_t^{(n)}(\omega)| \leq g_t(\omega)$ *for almost all* $\omega \in \Omega$ *and almost all* $t \in [a, b]$ *and all* n*; and that* $\mathbf{E}(g_t^2)$ *is Lebesgue integrable over* $[a, b]$.

Then f is GI-integrable on $[a, b]$ and that

$$\mathbf{E}\left((GI)\int_a^b (f^{(n)} - f)_t\ dB_t\right)^2 \to 0$$

as $n \to \infty$.

Proof. By Theorems 3.5.6 and 2.5.3, $\mathbf{E}(f_t^{(n)})^2$ is Lebesgue integrable on $[a,b]$. By Dominated Convergence Theorem for the Lebesgue integral, $\mathbf{E}(f_t)^2$ is integrable on $[a,b]$ and

$$(L)\int_a^b \mathbf{E}(f_t^{(n)} - f_t)^2 \, dt \to 0 \text{ as } n \to \infty.$$

Hence f is GI-integrable, by Theorem 3.5.6. By the isometric property (Theorem 3.6.2),

$$\mathbf{E}\Big((GI)\int_a^b (f^{(n)} - f)_t \, dB_t\Big)^2 = (L)\int_a^b \mathbf{E}(f^{(n)} - f)_t^2 \, dt.$$

Therefore

$$\mathbf{E}\Big((GI)\int_a^b (f^{(n)} - f)_t \, dB_t\Big)^2 \to 0$$

as $n \to \infty$. $\square$

Chapter 4

The Henstock-Young Integral

In Chapters 2 and 3, we define Stieltjes integrals of processes using L^1-norm or L^2-norm. In this chapter, we shall consider a Stieltjes integral of deterministic real-valued functions. In 1936, L.C.Young proved that the Riemann-Stieltjes integral $\displaystyle\int_a^b f\,dg$ exists, if $f \in BV_p, g \in BV_q, \frac{1}{p} + \frac{1}{q} > 1$ and f, g do not have common discontinuous points. In this chapter, using Henstock's approach, we prove that $\displaystyle\int_a^b f\,dg$ still exists without assuming condition on discontinuous points. Some convergence theorems are also proved. The integral considered here is useful for Stochastic Analysis, since most of the processes have paths of unbounded variation but have paths of bounded p-variation, where $p > 1$, see Section 4.6.

In this chapter, all δ-fine divisions are δ-fine Henstock divisions.

4.1 Definition of the HS-integral

In this section, we shall define an integral of Stieltjes-type using Henstock's approach.

Definition 4.1.1. Let $f, g : [a, b] \to \mathbb{R}$. Then f is said to be *Henstock-Stieltjes integrable* (or HS-integrable) to real number A on $[a, b]$ with respect to g if for every $\epsilon > 0$, there exists a positive function δ defined on $[a, b]$ such that for every δ-fine division $D = \{([t_i, t_{i+1}], \xi_i)\}_{i=1}^n$ of $[a, b]$, we have

$$|S(f, \delta, D) - A| \leq \epsilon,$$

where

$$S(f, \delta, D) = \sum_{i=1}^{n} f(\xi_i)(g(t_{i+1}) - g(t_i)).$$

We denote A by $(HS)\int_a^b f(t)\ dg(t)$ or $(HS)\int_a^b f\ dg$.

Theorem 4.1.2. *The integral A in the definition of HS-integral is unique.*

Proof. The proof is standard. Let $\epsilon > 0$ be given. Assume that A_1 and A_2 satisfy the condition in Definition 4.1.1, i.e., there exist δ_i, $i = 1, 2$, such that for every δ_i-fine divisions D_i of $[a, b]$, we have

$$|S(f, \delta_1, D_1) - A_1| \leq \frac{\epsilon}{2},$$

and

$$|S(f, \delta_2, D_2) - A_2| \leq \frac{\epsilon}{2}.$$

Pick $\delta = \min\{\delta_1, \delta_2\}$, then a δ-fine division of $[a, b]$ is also a δ_1-fine division and a δ_2-fine division of $[a, b]$. Then

$$\begin{aligned} |A_2 - A_1| =& |(S(f, \delta, D) - A_1) - (S(f, \delta, D) - A_2)| \\ \leq& |S(f, \delta, D) - A_1)| + |(S(f, \delta, D) - A_2)| \\ =& \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence, $A_1 = A_2$. □

Theorem 4.1.3. *Let $f, g : [a, b] \to \mathbb{R}$. Then f is HS-integrable on $[a, b]$ with respect to g if and only if there exist a real number A and a decreasing sequence of positive functions $\{\delta_n(\xi)\}_{n\in\mathbb{N}}$ defined on $[a, b]$ such that for every δ_n-fine divisions D_n of $[a, b]$, n=1,2,..., we have*

$$\lim_{n\to\infty} S(f, \delta_n, D_n) = A.$$

Proof. The proof is standard. Let $\epsilon > 0$ be given. Assume that f is HS-integrable to A on $[a, b]$ with respect to g, then, there exists a positive function δ on $[a, b]$ such that for every δ-fine division D of $[a, b]$, we have

$$|S(f, \delta, D) - A| \leq \epsilon.$$

Then, there exists a decreasing sequence of positive functions $\{\delta_n(\xi)\}_{n\in\mathbb{N}}$ such that for any δ_n-fine division D_n on $[a, b]$, we have

$$|S(f, \delta_n, D_n) - A| \leq \frac{1}{n},$$

thus, we can conclude that

$$\lim_{n\to\infty} |S(f, \delta_n, D_n) - A| = 0,$$

that is,

$$\lim_{n\to\infty} S(f, \delta_n, D_n) = A.$$

Conversely, assume that there exist a real number A and a positive decreasing sequence of positive functions $\{\delta_n(\xi)\}_{n\in\mathbb{N}}$ on $[a, b]$ such that for any δ_n-fine divisions $D_n, n = 1, 2, ...,$

$$\lim_{n\to\infty} S(f, \delta_n, D_n) - A = 0.$$

Suppose that f is not HS-integrable on $[a, b]$. Then there exists $\epsilon > 0$ such that for every positive function δ on $[a, b]$, there exists a δ-fine division D of $[a, b]$ with

$$|S(f, \delta, D) - A| \geq \epsilon.$$

Hence for each δ_n, there exists a δ_n-fine division D_n of $[a, b]$ such that

$$|S(f, \delta_n, D_n) - A| \geq \epsilon.$$

It contradicts that $\lim_{n\to\infty} S(f, \delta_n, D_n) = A$. Then we can conclude that f is HS-integrable. □

4.2 Basic properties

In this section, we shall prove some basic properties of the HS-integral and establish an integration by parts formula. All proofs of this section are standard.

Theorem 4.2.1. *Let $\alpha \in \mathbb{R}$. If $f, g : [a,b] \to \mathbb{R}$ are HS-integrable on $[a,b]$ with respect to h, then*

(i) $f + g$ is HS-integrable with respect to h, and

$$(HS)\int_a^b (f(t) + g(t))\ dh(t) = (HS)\int_a^b f(t)\ dh(t) + (HS)\int_a^b g(t)\ dh(t).$$

(ii) αf is HS-integrable with respect to h, and

$$(HS)\int_a^b (\alpha f(t))\ dh(t) = \alpha \cdot (HS)\int_a^b f(t)\ dh(t).$$

Proof. Let $\epsilon > 0$ and $\alpha \in \mathbb{R}$. Assume that f, g are HS-integrable functions on $[a,b]$ with respect to h such that

$$(HS)\int_a^b f(t)\ dh(t) = A,$$

and

$$(HS)\int_a^b g(t)\ dh(t) = B.$$

Then, there exist positive functions δ_1 and δ_2 on $[a,b]$ such that

$$|S(f, \delta_1, D_1) - A| \leq \frac{\epsilon}{2},$$

and

$$|S(g, \delta_2, D_2) - B| \leq \frac{\epsilon}{2}$$

for every δ_1, δ_2-fine divisions D_1, D_2 of $[a,b]$, respectively.

Pick $\delta(\xi) = \min\{\delta_1(\xi), \delta_2(\xi)\}$. Then, for every δ-fine division D of $[a,b]$, we have $|S(f, \delta, D) - A| \leq \frac{\epsilon}{2}$ and $|S(g, \delta, D) - B| \leq \frac{\epsilon}{2}$.

Since

$$S(f+g,\delta,D) = S(f,\delta,D) + S(g,\delta,D).$$

Then,

$$\begin{aligned}|S(f+g,\delta,D) - (A+B)| \leq &|S(f,\delta,D) - A| + |S(g,\delta,D) - B| \\ \leq &\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Hence, $f+g$ is HS-integrable function on $[a,b]$ with respect to h and

$$(HS)\int_a^b (f(t)+g(t))\ dh = (HS)\int_a^b f(t)\ dh + (HS)\int_a^b g(t)\ dh.$$

Next, consider (ii). It is obvious that

$$(HS)\int_a^b 0\cdot f(t)\ dh(t) = 0 = 0\cdot (HS)\int_a^b f(t)\ dh(t).$$

Assume that $\alpha \neq 0$, then there exists positive function δ on $[a,b]$ such that for every δ-fine division D of $[a,b]$,

$$|S(f,\delta,D) - A| \leq \frac{\epsilon}{\alpha},\ \text{where } A = \int_a^b f(t)\ dh(t).$$

Since

$$S(\alpha f,\delta,D) = \alpha S(f,\delta,D).$$

Then,

$$\begin{aligned}|S(\alpha f,\delta,D) - \alpha A| = &|\alpha S(f,\delta,D) - \alpha A| \\ = &\alpha|S(f,\delta,D) - A| \\ \leq &\alpha(\frac{\epsilon}{\alpha}) = \epsilon.\end{aligned}$$

Hence, αf is HS-integrable function on $[a,b]$ with respect to h, and

$$(HS)\int_a^b \alpha f\ dh = \alpha F = \alpha\cdot (HS)\int_a^b f\ dh.$$

$\square$

Theorem 4.2.2. *Let $a < c < b$. If f is HS-integrable on $[a, c]$ and $[c, b]$ with respect to g, then it is integrable on $[a, b]$ and*

$$(HS)\int_a^b f\ dg = (HS)\int_a^c f\ dg + (HS)\int_c^b f\ dg.$$

Proof. Let $\epsilon > 0$ be given. Let f is HS-integrable on $[a, c]$ and on $[c, b]$ with respect to g, with $F[u, v] = (HS)\int_u^v f\ dg$, then there exist positive functions δ_1 and δ_2 such that for any δ_1, δ_2-fine divisions D_1, D_2 of $[a, c]$ and $[c, b]$, respectively, we have

$$|S(f, \delta_1, D_1) - F[a, c]| \leq \frac{\epsilon}{2},$$

and

$$|S(f, \delta_2, D_2) - F[c, b]| \leq \frac{\epsilon}{2}.$$

Define

$$\delta(\xi) = \begin{cases} \min\{\delta_1(\xi), c - \xi\} & , \text{ if } \xi \in [a, c); \\ \min\{\delta_1(\xi), \delta_2(\xi)\} & , \text{ if } \xi = c; \\ \min\{\xi - c, \delta_2(\xi)\} & , \text{ if } \xi \in (c, b]. \end{cases}$$

Note that for any δ-fine division D of $[a, b]$, c is always a tag of D. Hence, for any δ-fine division D of $[a, b]$, we have $D = D_1 \cup D_2$, where D_i, $i = 1, 2$, are δ_i-fine divisions of $[a, c]$ and $[c, d]$, respectively. Thus we can see that

$$\begin{aligned} |S(f, \delta, D) - (F[a, c] + F[c, b])| &= |S(f, \delta_1, D_1) - F[a, c] + S(f, \delta_2, D_2) - F[c, b]| \\ &\leq |S(f, \delta_1, D_1) - F[a, c]| + |S(f, \delta_2, D_2) - F[c, b]| \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus f is HS-integrable on $[a, b]$ with respect to g, and

$$(HS)\int_a^b f\ dg = (HS)\int_a^c f\ dg + (HS)\int_c^b f\ dg.$$

□

Theorem 4.2.3 (Cauchy Criterion for HS-integrals)**.** *Let $f, g : [a, b] \to \mathbb{R}$. Then f is HS-integrable on $[a, b]$ with respect to g if and only if for every $\epsilon > 0$ there exists a positive function δ on $[a, b]$ such that for any two δ-fine divisions of $[a, b]$, $D = \{([t_i, t_{i+1}], \xi_i)\}$ and $D' = \{([t'_j, t'_{j+1}], \xi'_j)\}$, we have*

$$|S(f, \delta, D) - S(f, \delta, D')| \leq \epsilon.$$

Proof. Let $f, g : [a, b] \to \mathbb{R}$. Assume that f is HS-integrable on $[a, b]$ with respect to g.

Let $\epsilon > 0$ be given. Then, there exists a positive function δ on $[a, b]$ such that for any δ-fine divisions D and D' of $[a, b]$, we have

$$|S(f, \delta, D) - F| \leq \frac{\epsilon}{2},$$

and

$$|S(f, \delta, D') - F| \leq \frac{\epsilon}{2},$$

where $F = (HS)\displaystyle\int_a^b f\ dg$.

Thus, we can see that

$$\begin{aligned}|S(f, \delta, D) - S(f, \delta, D')| =& |(S(f, \delta, D) - F) - (S(f, \delta, D') - F)| \\ \leq& |S(f, \delta, D) - F| + |S(f, \delta, D') - F| \\ <& \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

and we have done the necessary condition.

Conversely, assume that for every $\epsilon > 0$, there exists a positive function δ on $[a, b]$ such that for any two δ-fine divisions of $[a, b]$, $D = \{([t_i, t_{i+1}], \xi_i)\}$ and $D' = \{([t'_j, t'_{j+1}], \xi'_j)\}$, we have

$$|S(f, \delta, D) - S(f, \delta, D')| \leq \frac{\epsilon}{2},$$

Let $\epsilon_n = \frac{2}{n}$, for $n \in \mathbb{N}$ and δ_n be the corresponding positive function on $[a, b]$. We may assume that if $m \geq n$, $\delta_m(\xi) \leq \delta_n(\xi)$ for each $\xi \in [a, b]$. Then a δ_m-fine division of $[a, b]$ is also a δ_n-fine division of $[a, b]$. Hence

$$|S(f,\delta_{m_1},D_{m_1}) - S(f,\delta_{m_2},D_{m_2})| \leq \frac{1}{n}$$

whenever $m_1, m_2 \geq n$.

We conclude that $\{S(f,\delta_n,D_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Therefore, there exists real number F such that $\lim_{n\to\infty}(|S(f,\delta_n,D_n) - F|) = 0$.

Let $\epsilon > 0$ be given. Then there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$,

$$|S(f,\delta_n,D_n) - F| \leq \frac{\epsilon}{2}.$$

Let $\frac{1}{N_2} \leq \frac{\epsilon}{2}$ and $N = max\{N_1, N_2\}$. Let $\delta(\xi) = \delta_N(\xi)$ for each $\xi \in [a,b]$. Then

$$\begin{aligned}
|S(f,\delta,D) - F| =& |S(f,\delta,D) - S(f,\delta_N,D_N)) + (S(f,\delta_N,D_N) - F)| \\
=& |S(f,\delta,D) - S(f,\delta_N,D_N) + S(f,\delta_N,D_N) - F| \\
\leq& |S(f,\delta,D) - S(f,\delta_N,D_N)| + |S(f,\delta_N,D_N) - F| \\
\leq& \frac{1}{N} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus f is HS-integrable on $[a,b]$. Moreover, we also know that $(HS)\int_a^b f\ dg = F$. □

Theorem 4.2.4. *Let $f,g : [a,b] \to \mathbb{R}$. If f is HS-integrable on $[a,b]$ with respect to g, then f is also HS-integrable on any subinterval $[c,d]$ of $[a,b]$ with respect to g.*

Proof. Let $\epsilon > 0$ be given and $[c,d]$ be subinterval of $[a,b]$. Assume that f is HS-integrable on $[a,b]$ with respect to g, then there exists a positive function δ on $[a,b]$ such that for any two δ-fine divisions D_1, D_2 of $[a,b]$,

$$|S(f,\delta,D_1) - S(f,\delta,D_2)| \leq \epsilon.$$

Let D', D'' be δ-fine divisions of $[c,d]$ and D be δ-fine division of $[a,c] \cup [c,d]$. Then $D \cup D'$ and $D \cup D''$ form δ-fine divisions of $[a,b]$. Hence, we can see that

$$|S(f,\delta,D') - S(f,\delta,D'')| = |S(f,\delta,D \cup D') - S(f,\delta,D \cup D'')| \leq \epsilon.$$

From Theorem 4.2.3, we can conclude that f is HS-integrable on $[c,d]$. $\square$

Lemma 4.2.5 (Henstock's lemma for HS-integral). *Let $f:[a,b]\to\mathbb{R}$ be HS-integrable on $[a,b]$ with respect to g. Let $F(u,v)=(HS)\int_u^v f\ dg$ for any $[u,v]\subseteq[a,b]$. Then for every $\epsilon>0$, there exists a positive function δ on $[a,b]$ such that for every δ-fine partial division $D'=\{([u_i,v_i],\xi_i)\}$ of $[a,b]$, we have*

$$(D')\sum|f(\xi_i)(g(v_i)-g(u_i))-F[u_i,v_i]|\le\epsilon.$$

Proof. Let $\epsilon>0$ be given. Let $f:[a,b]\to\mathbb{R}$ be HS-integrable on $[a,b]$. Then there exists a positive function δ on $[a,b]$ such that for every δ-fine division $D=\{([t_k,t_{k+1}],\xi_k)\}_{k=1}^m$ of $[a,b]$, we have

$$|(D)\sum_{k=1}^m f(\xi_k)(g(t_{k+1})-g(t_k))-F(t_k,t_{k+1})|\le\frac{\epsilon}{4}.$$

Let $D'=\{([u_i,v_i],\xi_i)\}_{i=1}^n$ be a δ-fine partial division of $[a,b]$. The set $[a,b]\setminus\cup_{i=1}^n[u_i,v_i]$ consists of a finite number of disjoint intervals. Let $K_j, 1\le j\le p$ be the closures of these subintervals. Since K_j is a subinterval of $[a,b]$ then there exists δ-fine division D_j of K_j such that

$$|S(f,\delta,D_j)-(HS)\int_{K_j}f\ dg|\le\frac{\epsilon}{4p}.$$

Then $D=D'\cup(\cup_{j=1}^p D_j)$ is a δ-fine division of $[a,b]$,

$$|(S(f,\delta,D')-(HS)\int_{\cup_{i=1}^n[u_i,v_i]}f\ dg)+\sum_{j=1}^p(S(f,\delta,D_j)-(HS)\int_{K_j}f\ dg)|$$

$$=|S(f,\delta,D)-(HS)\int_{[a,b]}f\ dg|\le\frac{\epsilon}{4}.$$

Hence,

$$|S(f,\delta,D')-(HS)\int_{\cup_{i=1}^n[u_i,v_i]}f\ dg|$$

$$<\frac{\epsilon}{4}+|(\sum_{j=1}^p(S(f,\delta,D_j)-(HS)\int_{K_j}f\ dg))|\le\frac{\epsilon}{4}+p\frac{\epsilon}{4p}=\frac{\epsilon}{2}.$$

Then, for any δ-fine partial division D' of $[a,b]$,

$$|(D')\sum f(\xi_i)(g(v_i)-g(u_i))-F(u_i,v_i)|\leq\frac{\epsilon}{2}.$$

Now, let $D'=\{([u_i,v_i],\xi_i)\}$ be any δ-fine partial division.

Let

$$D'_+=\{([u_i,v_i],\xi_i)\in D': f(\xi_i)(g(v_i)-g(u_i))-F(u_i,v_i)\geq 0\},$$

and

$$D'_-=\{([u_i,v_i],\xi_i)\in D': f(\xi_i)(g(v_i)-g(u_i))-F(u_i,v_i)< 0\}.$$

Since D_+ and D_- are δ-fine partial divisions of $[a,b]$, then

$$|(D'_+)\sum_{i=1}^{n} f(\xi_i)(g(v_i)-g(u_i))-F(u_i,v_i)|\leq\frac{\epsilon}{2},$$

and

$$|(D'_-)\sum_{i=1}^{n} f(\xi_i)(g(v_i)-g(u_i))-F(u_i,v_i)|\leq\frac{\epsilon}{2}.$$

Hence, we can see that

$$\begin{aligned}
&(D')\sum_{i=1}^{n}|f(\xi_i)(g(v_i)-g(u_i))-F(u_i,v_i)|\\
=&(D'_+)\sum_{i=1}^{n} f(\xi_i)(g(v_i)-g(u_i))-F(u_i,v_i)-(D'_-)\sum_{i=1}^{n} f(\xi_i)(g(v_i)-g(u_i))-F(u_i,v_i)\\
=&|(D'_+)\sum_{i=1}^{n} f(\xi_i)(g(v_i)-g(u_i))-F(u_i,v_i)|+|(D'_-)\sum_{i=1}^{n} f(\xi_i)(g(v_i)-g(u_i))-F(u_i,v_i)|\\
\leq&\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}$$

□

Theorem 4.2.6 (Integration by parts)**.** *Let $f,g:[a,b]\to\mathbb{R}$. If f is HS-integrable on $[a,b]$ with respect to g and for every $\epsilon>0$, there exists a positive function δ such that for any δ-fine partial division $D=\{([u_i,v_i],\xi_i)\}$,*

$$|(D)\sum(f(v_i)-f(u_i))(g(v_i)-g(u_i))|\leq\epsilon$$

then g is HS-integrable on $[a, b]$ with respect to f, and

$$(HS)\int_a^b g\ df = f(b)g(b) - f(a)g(a) - (HS)\int_a^b f\ dg.$$

Proof. Let $\epsilon > 0$ be given and let $f, g : [a, b] \to \mathbb{R}$. Assume that f is HS-integrable to A on $[a, b]$ with respect to g, then there exists a positive function δ_1 on $[a, b]$ such that for any δ_1-fine partial division $D' = \{([u_i, v_i], \xi_i)\}$ of $[a, b]$,

$$|(D')\sum(f(v_i) - f(u_i))(g(v_i) - g(u_i))| \leq \frac{\epsilon}{3}.$$

Since f is HS-integrable to A on $[a, b]$ with respect to g then there exists a positive function δ_2 on $[a, b]$ such for any δ_2-fine division $D'' = \{([t_i, t_{i+1}], \xi_i)\}$ of $[a, b]$, we have

$$|(D'')\sum f(\xi_i)(g(t_{i+1}) - g(t_i)) - A| \leq \frac{\epsilon}{3}.$$

Choose $\delta(\xi) = \min\{\delta_1(\xi), \delta_2(\xi)\}$. Let $D = \{([t_i, t_{i+1}], \xi_i)\}$ be a δ-fine partial division of $[a, b]$. In the following, let $F(u, v) = (HS)\int_u^v f\ dg$. We can see that

$$\begin{aligned}
&|((D)\textstyle\sum g(\xi_i)(f(t_{i+1}) - f(t_i))) - (f(b)g(b) - f(a)g(a) - A)| \\
=&|(D)\sum\Big(g(\xi_i)(f(t_{i+1}) - f(t_i)) - f(t_{i+1})g(t_{i+1}) + f(t_i)g(t_i) + F[t_i, t_{i+1}]\Big)| \\
=&|(D)\sum -f(\xi_i)(g(t_{i+1}) - g(t_i)) + (f(\xi_i) - f(t_i))(g(\xi_i) - g(t_i)) \\
&\quad - (f(t_{i+1}) - f(\xi_i))(g(t_{i+1}) - g(\xi_i)) + F[t_i, t_{i+1}]| \\
\leq&|(D)\sum f(\xi_i)(g(t_{i+1}) - g(t_i)) - F[t_i, t_{i+1}]| \\
&\quad + |(D)\sum(f(\xi_i) - f(t_i))(g(\xi_i) - g(t_i))| + |(D)\sum(f(t_{i+1}) - f(\xi_i))(g(t_{i+1}) - g(\xi_i))| \\
<&\frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Thus, we can conclude that g is HS-integrable to $f(b)g(b) - f(a)g(a) - A$ on $[a, b]$ with respect to f.

Hence, we have

$$(HS)\int_a^b f\ dg = f(b)g(b) - f(a)g(a) - (HS)\int_a^b g\ df.$$

$\square$

4.3 Young-Love inequality

In this section, we shall present some results proved by L.C.Young (1936), see [13, 24].

Lemma 4.3.1. *If $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ are two finite sequences of real numbers, and $p, q > 0$, then there is an index k $(0 < k \leq n)$, such that*

$$|a_k b_k| \leq \big[\frac{1}{n}\sum_{i=1}^{n}|a_i|^p\big]^{1/p}\big[\frac{1}{n}\sum_{i=1}^{n}|b_i|^q\big]^{1/q}.$$

Proof. Let $c = (c_1, c_2, ..., c_n)$ be a finite sequence of positive real numbers. We have

$$c_1 c_2 ... c_n \leq \big[\frac{1}{n}\sum_{i=1}^{n} c_i\big]^n. \tag{4.3.2}$$

Let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ be two finite sequences of real numbers, and $p, q > 0$. Let

$$|a_k b_k| = \min\{|a_1 b_1|, |a_2 b_2|, ..., |a_n b_n|\}. \tag{4.3.3}$$

Then, by (4.3.2) and (4.3.3), we can see that

$$\begin{aligned}|a_k b_k| \leq & |(a_1 b_1)(a_2 b_2)...(a_n b_n)|^{1/n} \\ = & \big[(|a_1|^p|a_2|^p...|a_n|^p)^{1/n}\big]^{1/p}\big[(|b_1|^q|b_2|^q...|b_n|^q)^{1/n}\big]^{1/q} \\ \leq & \big[\frac{1}{n}\sum_{i=1}^{n}|a_i|^p\big]^{1/p}\big[\frac{1}{n}\sum_{i=1}^{n}|b_i|^q\big]^{1/q}.\end{aligned}$$

□

Lemma 4.3.4 (Hölder inequality)**.** *If $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ are two finite sequences of real numbers, and $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} \geq 1$, then*

$$\sum_{i=1}^{n}|a_i b_i| \leq \big[\sum_{i=1}^{n}|a_i|^p\big]^{1/p}\big[\sum_{i=1}^{n}|b_i|^q\big]^{1/q}.$$

Proof. Let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ be two finite sequences of real numbers, and $p, q > 0$. First, we shall consider the case where $\frac{1}{p} + \frac{1}{q} > 1$ suppose $|a_i b_i|$ be arranged in decreasing order. Then, by Lemma 4.3.1, we have

$$|a_n b_n| \leq n^{-(\frac{1}{p}+\frac{1}{q})} \big[\sum_{i=1}^{n} |a_i|^p\big]^{1/p} \big[\sum_{i=1}^{n} |b_i|^q\big]^{1/q}.$$

Similarly,

$$\begin{aligned}|a_{n-1} b_{n-1}| \leq& (n-1)^{-(\frac{1}{p}+\frac{1}{q})} \big[\sum_{i=1}^{n-1} |a_i|^p\big]^{1/p} \big[\sum_{i=1}^{n-1} |b_i|^q\big]^{1/q} \\ \leq& (n-1)^{-(\frac{1}{p}+\frac{1}{q})} \big[\sum_{i=1}^{n} |a_i|^p\big]^{1/p} \big[\sum_{i=1}^{n} |b_i|^q\big]^{1/q}.\end{aligned}$$

Finally, by proceeding in this way, we can see that

$$\sum_{i=1}^{n} |a_i b_i| \leq \zeta(\frac{1}{p} + \frac{1}{q}) \big[\sum_{i=1}^{n} |a_i|^p\big]^{1/p} \big[\sum_{i=1}^{n} |b_i|^q\big]^{1/q},$$

where $\zeta(\frac{1}{p} + \frac{1}{q}) = \sum_{i=1}^{\infty} i^{-(\frac{1}{p}+\frac{1}{q})}$.

Next, consider

$$\begin{aligned}\big[\sum_{i=1}^{n} |a_i b_i|\big]^2 =& \sum_{i=1}^{n} \sum_{j=1}^{n} |a_i a_j b_i b_j| \\ \leq& \zeta(\frac{1}{p} + \frac{1}{q}) \big[\sum_{i=1}^{n} \sum_{j=1}^{n} |a_i a_j|^p\big]^{1/p} \big[\sum_{i=1}^{n} \sum_{j=1}^{n} |b_i b_j|^q\big]^{1/q} \\ =& \zeta(\frac{1}{p} + \frac{1}{q}) \big[\big(\sum_{i=1}^{n} |a_i|^p\big)^2\big]^{1/p} \big[\big(\sum_{i=1}^{n} |b_i|^q\big)^2\big]^{1/q}.\end{aligned}$$

Hence, we can reduce the above inequality to

$$\sum_{i=1}^{n} |a_i b_i| \leq \sqrt{\zeta(\frac{1}{p} + \frac{1}{q})} \big[\sum_{i=1}^{n} |a_i|^p\big]^{1/p} \big[\sum_{i=1}^{n} |b_i|^q\big]^{1/q}.$$

Proceeding in this way, we have

$$\sum_{i=1}^{n} |a_i b_i| \leq \big(\zeta(\frac{1}{p} + \frac{1}{q})\big)^{2^{-N}} \big[\sum_{i=1}^{n} |a_i|^p\big]^{1/p} \big[\sum_{i=1}^{n} |b_i|^q\big]^{1/q}.$$

$\left(\zeta(\frac{1}{p}+\frac{1}{q})\right)^{2^{-N}} \to 1$, as $N \to \infty$. Then we have

$$\sum_{i=1}^{n} |a_i b_i| \leq \Big[\sum_{i=1}^{n} |a_i|^p\Big]^{1/p} \Big[\sum_{i=1}^{n} |b_i|^q\Big]^{1/q}. \tag{4.3.5}$$

Now, consider $\frac{1}{p_1}+\frac{1}{q_1}=1$. Let $p_n \to p_1$, $q_n \to q_1$, $p_n, q_n > 0$ and $\frac{1}{p_n}+\frac{1}{q_n} > 1$, then (4.3.5) holds for $p = p_n$ and $q = q_n$. Let $n \to \infty$ in (4.3.5), we get the result for the case $\frac{1}{p_1}+\frac{1}{q_1}=1$. □

Definition 4.3.6. Let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ be two finite sequences of real numbers. If $a = (a_1, a_2, ..., a_n)$ is partitioned into m parts, and in the k^{th} part, we have $(a_i, a_{i+1}, ..., a_j)$, then we let

$$x_k = \sum_{l=i}^{j} a_l.$$

Hence, we have a new sequence $x = (x_1, x_2, ..., x_m)$. Similarly, we partition $b = (b_1, b_2, ..., b_n)$ into m parts, then we get a new sequence $y = (y_1, y_2, ..., y_m)$. Thus, we have

$$\Big[\sum_{i=1}^{m} |x_i|^p\Big]^{\frac{1}{p}} \Big[\sum_{i=1}^{m} |y_i|^q\Big]^{\frac{1}{q}}.$$

There are only finite numbers of ways to partition a and b.

Let $S_{p,q}(a,b)$ be the maximum of

$$\Big[\sum_{i=1}^{m} |x_i|^p\Big]^{\frac{1}{p}} \Big[\sum_{i=1}^{m} |y_i|^q\Big]^{\frac{1}{q}},$$

where the maximum is over all possible x and y.

Lemma 4.3.7 (Young-Love inequality)**.** *If $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ are two finite sequences of real numbers, and $p, q > 0$ and $\frac{1}{p}+\frac{1}{q} \geq 1$, then*

$$\left|\sum_{j=1}^{n} \Big[\sum_{i=1}^{j} a_i\Big] b_j\right| \leq \left\{1 + \zeta(\frac{1}{p}+\frac{1}{q})\right\} S_{p,q}(a,b),$$

where $\zeta(\frac{1}{p}+\frac{1}{q}) = \sum_{n=1}^{\infty} n^{-(\frac{1}{p}+\frac{1}{q})}$.

Proof. Let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ be two finite sequences of real numbers, and $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} \geq 1$. Let k be a fixed integer where $0 < k \leq n-1$. Define x_i, y_i by

$$x_i, y_i = \begin{cases} a_i, b_i & , \text{ if } i < k; \\ a_i + a_{i+1}, b_i + b_{i+1} & , \text{ if } i = k; \\ a_{i+1}, b_{i+1} & , \text{ if } k < i \leq n-1, \end{cases}$$

respectively.

Let $x = (x_1, x_2, ..., x_{n-1})$ and $y = (y_1, y_2, ..., y_{n-1})$. Then, we have

$$\begin{aligned} \sum_{j=1}^{n-1} \Big[\sum_{i=1}^{j} x_i\Big] y_j =& \sum_{j=1}^{k-1} \Big[\sum_{i=1}^{j} a_i\Big] b_j + (a_1 + a_2 + ... + a_{k+1})(b_k + b_{k+1}) \\ &+ \sum_{j=k+1}^{n-1} \Big[\sum_{i=1}^{j} a_{i+1}\Big] b_{j+1} \\ =& a_{k+1} b_k + \sum_{j=1}^{n} \Big[\sum_{i=1}^{j} a_i\Big] b_j. \end{aligned}$$

By Lemma 4.3.1, there is k, $(0 < k \leq n-1)$, such that

$$|a_{k+1} b_k| \leq \Big[\frac{1}{n-1} \sum_{i=1}^{n-1} |a_{i+1}|^p\Big]^{1/p} \Big[\frac{1}{n-1} \sum_{i=1}^{n-1} |b_i|^q\Big]^{1/q} \leq (n-1)^{-(\frac{1}{p}+\frac{1}{q})} S_{p,q}(a, b).$$

Similarly, there is k (maybe different from the above k) such that

$$\begin{aligned} |x_{k+1} y_k| \leq& \Big[\frac{1}{n-2} \sum_{i=1}^{n-2} |x_{i+1}|^p\Big]^{1/p} \Big[\frac{1}{n-2} \sum_{i=1}^{n-2} |y_i|^q\Big]^{1/q} \\ \leq& (n-2)^{-(\frac{1}{p}+\frac{1}{q})} S_{p,q}(x, y) \leq (n-2)^{-(\frac{1}{p}+\frac{1}{q})} S_{p,q}(a, b). \end{aligned}$$

Hence,

$$\begin{aligned}
\Big|\sum_{j=1}^{n}\big[\sum_{i=1}^{j}a_i\big]b_j\Big| \leq& |a_{k+1}b_k| + \Big|\sum_{j=1}^{n-1}\big[\sum_{i=1}^{j}x_i\big]y_j\Big| \\
\leq& (n-1)^{-(\frac{1}{p}+\frac{1}{q})}S_{p,q}(a,b) + \Big|\sum_{j=1}^{n-1}\big[\sum_{i=1}^{j}x_i\big]y_j\Big| \\
\leq& \Big\{\big[(n-1)^{-(\frac{1}{p}+\frac{1}{q})} + (n-2)^{-(\frac{1}{p}+\frac{1}{q})} + ... + 1\big] + 1\Big\}S_{p,q}(a,b) \\
\leq& \Big\{1+\zeta(\frac{1}{p}+\frac{1}{q})\Big\}S_{p,q}(a,b),
\end{aligned}$$

where $\zeta(\frac{1}{p}+\frac{1}{q}) = \sum_{n=1}^{\infty} n^{-(\frac{1}{p}+\frac{1}{q})}$. □

Definition 4.3.8. Let f be a real valued function defined on $[a,b]$ and let $0<p<\infty$. Given a partition $D=\{[t_i,t_{i+1}]\}_{i=1}^{n}$ of $[a,b]$, let

$$V_p(f,D,[a,b]) := \Big[\sum_{i=1}^{n}\big|f(t_{i+1})-f(t_i)\big|^p\Big]^{1/p}.$$

The p-variation of f is defined by

$$V_p(f,[a,b]) = \sup_D V_p(f,D,[a,b]),$$

where supremum is over all partition D. We say that $f\in BV_p[a,b]$ if $V_p(f,[a,b])<\infty$.

In this chapter, $V_p(f,[a,b])$ and $V_p(f,D,[a,b])$ are always denoted by $V_p(f)$ and $V_p(f,D)$, respectively.

Theorem 4.3.9. *Let $f\in BV_p[a,b]$ and $g\in BV_q[a,b]$, with $p,q>0$ and $\frac{1}{p}+\frac{1}{q}\geq 1$. Then, for any partition $D=\{[t_i,t_{i+1}]\}_{i=1}^{n}$ of $[a,b]$,*

$$\Big|\sum_{i=1}^{n} f(t_{i+1})\big(g(t_{i+1})-g(t_i)\big) - f(a)\big(g(b)-g(a)\big)\Big| \leq \Big\{1+\zeta(\frac{1}{p}+\frac{1}{q})\Big\}V_p(f)V_q(g),$$

where $\zeta\big(\frac{1}{p}+\frac{1}{q}\big) = \sum_{n=1}^{\infty} n^{-(\frac{1}{p}+\frac{1}{q})}$.

Proof. From Abel's transformation, we have

$$\sum_{i=1}^{n} f(t_{i+1})(g(t_{i+1})-g(t_i)) = \sum_{j=1}^{n}\Big[\sum_{i=1}^{j}(f(t_{i+1})-f(t_i))\Big](g(t_{j+1})-g(t_j))+f(a)(g(b)-g(a)).$$

Applying Lemma 4.3.7 with $a_i = f(t_{i+1}) - f(t_i)$, $b_j = g(t_{j+1}) - g(t_j)$, $\bar{a} = (a_1, a_2, ..., a_n)$ and $\bar{b} = (b_1, b_2, ..., b_n)$, we know that

$$\begin{aligned}
&\Big|\sum_{i=1}^{n} f(t_{i+1})\big(g(t_{i+1}) - g(t_i)\big) - f(a)\big(g(b) - g(a)\big)\Big| \\
&\qquad =\Big|\sum_{j=1}^{n}\Big[\sum_{i=1}^{j}(f(t_{i+1}) - f(t_i))\Big](g(t_{j+1}) - g(t_j))\Big| \\
&\qquad \le\Big\{1 + \zeta\big(\frac{1}{p} + \frac{1}{q}\big)\Big\}S_{p,q}(\bar{a}, \bar{b}) \\
&\qquad \le\Big\{1 + \zeta\big(\frac{1}{p} + \frac{1}{q}\big)\Big\}V_p(f; [a, b])V_q(g; [a, b]),
\end{aligned}$$

where $\zeta\big(\frac{1}{p} + \frac{1}{q}\big) = \sum_{n=1}^{\infty} n^{-(\frac{1}{p}+\frac{1}{q})}$. □

In the above, we have used the fact that $\sum_{i=r}^{s} a_i = f(t_{s+1}) - f(t_r)$.

Remark 4.3.10. *From inequality in Theorem 4.3.9, by changing the sign of the variable, we can prove that*

$$\Big|\sum_{i=1}^{n} f(t_{i+1})\big(g(t_{i+1})-g(t_i)\big)-f(b)\big(g(b)-g(a)\big)\Big| \le \Big\{1+\zeta\big(\frac{1}{p}+\frac{1}{q}\big)\Big\}V_p(f; [a, b])V_q(g; [a, b]).$$

Hence from these two inequalities, we also know that

$$\Big|\sum_{i=1}^{n} f(t_{i+1})\big(g(t_{i+1})-g(t_i)\big)-f(\xi)\big(g(b)-g(a)\big)\Big| \le \Big\{1+\zeta\big(\frac{1}{p}+\frac{1}{q}\big)\Big\}V_p(f; [a, b])V_q(g; [a, b]),$$

for any $\xi = t_i$ for some i.

Corollary 4.3.11. *Let $f \in BV_p[a, b]$ and $g \in BV_q[a, b]$, with $p, q > 0$ and $\frac{1}{p}+\frac{1}{q} > 1$. Then, for any two divisions $D = \{([t_i, t_{i+1}], \xi_i)\}_{i=1}^{n}$ and $D' = \{([s_j, s_{j+1}], \eta_j)\}_{j=1}^{m}$ of*

$[a,b]$, we have

$$\Big|(D)\sum_{i=1}^{n} f(\xi_i)\big(g(t_{i+1})-g(t_i)\big)-(D')\sum_{j=1}^{m} f(\eta_j)\big(g(s_{j+1})-g(s_j)\big)\Big| \le 2\Big\{1+\zeta\big(\frac{1}{p}+\frac{1}{q}\big)\Big\}V_p(f)V_q(g),$$

where $\zeta\big(\frac{1}{p}+\frac{1}{q}\big)=\sum_{n=1}^{\infty} n^{-(\frac{1}{p}+\frac{1}{q})}$.

Proof. Let $f \in BV_p[a,b]$ and $g \in BV_q[a,b]$, with $p,q>0$ and $\frac{1}{p}+\frac{1}{q}>1$. Let $D=\{([t_i,t_{i+1}],\xi_i)\}_{i=1}^{n}$ and $D'=\{([s_j,s_{j+1}],\eta_j)\}_{j=1}^{m}$ be two divisions of $[a,b]$ and $D''=\{[u_k,v_k]\}_{k=1}^{l}$ be the refined partition of $\{[t_i,\xi_i]\}_{i=1}^{n}$, $\{[\xi_i,t_{i+1}]\}_{i=1}^{n}$ and $\{[s_j,\eta_j]\}_{j=1}^{m}$, $\{[\eta_j,s_{j+1}]\}_{j=1}^{m}$.

From the remark above, we have

$$\begin{aligned}
&\Big|\sum_{k=1}^{l} f(v_k)\big(g(v_k)-g(u_k)\big)-(D)\sum_{i=1}^{n} f(\xi_i)\big(g(t_{i+1})-g(t_i)\big)\Big| \\
&\qquad\le \Big\{1+\zeta\big(\frac{1}{p}+\frac{1}{q}\big)\Big\}\sum_{i=1}^{n} V_p(f;[t_i,t_{i+1}])V_q(g;[t_i,t_{i+1}]) \\
&\qquad\le \Big\{1+\zeta\big(\frac{1}{p}+\frac{1}{q}\big)\Big\}\Big(\sum_{i=1}^{n} V_p^p(f;[t_i,t_{i+1}])\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^{n} V_q^q(g;[t_i,t_{i+1}])\Big)^{\frac{1}{q}} \\
&\qquad\le \Big\{1+\zeta\big(\frac{1}{p}+\frac{1}{q}\big)\Big\}V_p(f;[a,b])V_q(g;[a,b]).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\Big|(D)\sum_{i=1}^{n} f(\xi_i)\big(g(t_{i+1})-g(t_i)\big)-(D')\sum_{j=1}^{m} f(\eta_j)\big(g(s_{j+1})-g(s_j)\big)\Big|\\
&\qquad\le\Big|\sum_{k=1}^{l} f(v_k)\big(g(v_k)-g(u_k)\big)-(D)\sum_{i=1}^{n} f(\xi_i)\big(g(t_{i+1})-g(t_i)\big)\Big|\\
&\qquad\quad+\Big|\sum_{k=1}^{l} f(v_k)\big(g(v_k)-g(u_k)\big)-(D')\sum_{j=1}^{m} f(\eta_j)\big(g(s_{j+1})-g(s_j)\big)\Big|\\
&\qquad\le\Big\{1+\zeta\big(\frac{1}{p}+\frac{1}{q}\big)\Big\}\sum_{i=1}^{n} V_p(f;[t_i,t_{i+1}])V_q(g;[t_i,t_{i+1}])\\
&\qquad\quad+\Big\{1+\zeta\big(\frac{1}{p}+\frac{1}{q}\big)\Big\}\sum_{j=1}^{m} V_p(f;[s_j,s_{j+1}])V_q(g;[s_j,s_{j+1}])\\
&\qquad\le 2\Big\{1+\zeta\big(\frac{1}{p}+\frac{1}{q}\big)\Big\}V_p(f)V_q(g),
\end{aligned}$$

where $\zeta\big(\frac{1}{p}+\frac{1}{q}\big)=\sum_{n=1}^{\infty} n^{-(\frac{1}{p}+\frac{1}{q})}$. □

Lemma 4.3.12 (Jensen's inequality). *Let $a=(a_1,a_2,...,a_n)$ be a finite sequence of real numbers. If $0<p<p_1$ then*

$$\Big[\sum_{i=1}^{n}|a_i|^{p_1}\Big]^{1/p_1}\le\Big[\sum_{i=1}^{n}|a_i|^{p}\Big]^{p}$$

Proof. Let $a=(a_1,a_2,...,a_n)$ be a finite sequence of real numbers.

Define $b=\{b_i\}_{i=1}^{n}$ by

$$b_i=\frac{a_i}{\Big[\sum_{i=1}^{n}|a_i|^p\Big]^{1/p}}.$$

Then, we can see that

$$\sum_{i=1}^{n}|b_i|^p=\sum_{i=1}^{n}\left|\frac{a_i}{\Big[\sum_{i=1}^{n}|a_i|^p\Big]^{1/p}}\right|^p=1,$$

and

$$|b_i|=\left|\frac{a_i}{\Big[\sum_{i=1}^{n}|a_i|^p\Big]^{1/p}}\right|\le 1 \text{ for every } i=1,2,...,n.$$

Hence, $\sum_{i=1}^{n} |b_i|^{p_1} \leq \sum_{i=1}^{n} |b_i|^{p} = 1$, that is,

$$\frac{\Big[\sum_{i=1}^{n} |a_i|^{p_1}\Big]^{1/p_1}}{\Big[\sum_{i=1}^{n} |a_i|^{p}\Big]^{1/p}} = \left[\sum_{i=1}^{n} \left|\frac{a_i}{\Big[\sum_{i=1}^{n} |a_i|^{p}\Big]^{1/p}}\right|^{p_1}\right]^{1/p_1} \leq 1^{1/p_1} = 1.$$

Finally, we can conclude that

$$\Big[\sum_{i=1}^{n} |a_i|^{p_1}\Big]^{1/p_1} \leq \Big[\sum_{i=1}^{n} |a_i|^{p}\Big]^{1/p}.$$

□

Theorem 4.3.13. *If $f \in BV_p[a,b]$ and $0 < p < p_1$, then $f \in BV_{p_1}[a,b]$.*

Proof. Let $f \in BV_p[a,b]$. From Jensen's inequality, we have

$$\Big[\sum |f(v_i) - f(u_i)|^{p_1}\Big]^{1/p_1} \leq \Big[\sum |f(v_i) - f(u_i)|^{p}\Big]^{1/p},$$

for any partition $D = \{[u_i, v_i]\}$ of $[a,b]$.

Hence,

$$V_{p_1}(f) \leq V_p(f) < \infty,$$

therefore, $f \in BV_p[a,b]$. □

4.4 Integrable functions

In this section, we shall prove that if $f \in BV_p[a,b]$ and $g \in BV_q[a,b]$, where $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} > 1$, then f is HS-integrable with respect to g on $[a,b]$

Definition 4.4.1. Let $f : [a,b] \to \mathbb{R}$. Then f is said to be regulated if f has one-sided limits at every point of $[a,b]$, i.e., $\lim_{t \to c^+} f(t)$ and $\lim_{t \to c^-} f(t)$ exist, for each $c \in [a,b]$. The set of all regulated functions defined on $[a,b]$ is denoted by $RF[a,b]$.

Lemma 4.4.2. *If $f : [a, b] \to \mathbb{R}$ is regulated, then for every $\epsilon > 0$, there is a partition $D = \{[t_i, t_{i+1}]\}_{i=1}^n$ such that for each $i = 1, 2, .., n$, whenever $\xi, \eta \in (t_i, t_{i+1})$, we have*

$$|f_\xi - f_\eta| < \epsilon. \tag{4.4.3}$$

Proof. The following proof is given in [1]. Let $\epsilon > 0$ be given. Let $\mathcal{D}$ be the set of all $\zeta \in (a, b]$ such that there is a finite sequence $a = t_1 < t_2 < ... < t_{k+1} = \zeta$ satisfying 4.4.3 for $i = 1, 2, ..., k + 1$.

Since $f(a^+) = \lim_{t \to a^+} f(t)$ exists, there is $\zeta > a$ such that for any $t \in (a, \zeta)$

$$|f(t) - f(a^+)| < \frac{\epsilon}{2}.$$

Then, for any $t', t'' \in (a, \zeta)$

$$|f(t') - f(t'')| < |f(t') - f(a^+)| + |f(t'') - f(a^+)| < \epsilon.$$

Hence, $\zeta \in \mathcal{D}$. Thus $\mathcal{D}$ is nonempty. Let $d = \sup \mathcal{D}$.

We shall prove that $d \in \mathcal{D}$. Since $f(d^-) = \lim_{t \to d^-} f(t)$ exists, there is $\delta > 0$ such that for every $t \in (d - \delta, d)$, $|f(t) - f(d^-)| \leq \frac{\epsilon}{2}$. Let $\zeta \in \mathcal{D} \cap (d - \delta, d)$. Since $\zeta \in \mathcal{D}$, then there is a finite sequence $a = t_1 < t_2 < ... < t_{k+1} = \zeta$ such that (4.4.3) holds for $i = 1, 2, ..., k + 1$. Denote $t_{k+2} = d$, then, for any $t', t'' \in (\zeta, d) = (\zeta, t_{k+2})$

$$|f(t') - f(t'')| \leq |f(t') - f(d^-)| + |f(t'') - f(d^-)| \leq \epsilon.$$

Hence, $d \in \mathcal{D}$. Suppose $d \neq b$, i.e., $d < b$. Since $f(d^+) = \lim\limits_{t \to d^+} f(t)$ exists, there is $x > d$ with $x < b$ such that for any $t \in (d, x)$

$$|f(t) - f(d^+)| \leq \frac{\epsilon}{2}.$$

Then, for any $t', t'' \in (d, x)$

$$|f(t') - f(t'')| \leq |f(t') - f(d^+)| + |f(t'') - f(d^+)| \leq \epsilon.$$

Hence, $x \in \mathcal{D}$, it contradicts that $d = \sup\{\mathcal{D}\}$. So, $d = b$. $\square$

The following result is known, for example, see [1, p.24]. However we shall give a proof here

Theorem 4.4.4. *If s is a step function and $g \in RF[a,b]$, then s is HS-integrable with respect to g on $[a,b]$.*

Proof. We only prove the following case. Let s be a step function defined by

$$s(t) = \begin{cases} C_1\ , & \text{if } t = a; \\ C_2\ , & \text{if } a < t < c; \\ C_3\ , & \text{if } t = c; \\ C_4\ , & \text{if } c < t < b; \\ C_5\ , & \text{if } t = b. \end{cases}$$

We shall prove that s is HS-integrable with respect to g on $[a,b]$ and

$$\begin{aligned}(HS)\int_a^b s\ dg =& C_1(g(a^+) - g(a)) + C_2(g(c^-) - g(a^+)) + C_3(g(c^+) - g(c^-)) \\ &+ C_4(g(b^-) - g(c^+)) + C_5(g(b) - g(b^-)),\end{aligned}$$

where

$$g(a^+) = \lim_{t\to a^+} g(t) \text{ and } g(a^-) = \lim_{t\to a^-} g(t).$$

Let $\epsilon > 0$ be given. Then there exists $\delta_1 > 0$, such that

$$|g(t) - g(a^+)| < \frac{\epsilon}{8C_m} \text{ whenever } 0 < t - a < \delta_1,$$

$$|g(t) - g(c^-)| < \frac{\epsilon}{8C_m} \text{ whenever } 0 < c - t < \delta_1,$$

$$|g(t) - g(c^+)| < \frac{\epsilon}{8C_m} \text{ whenever } 0 < t - c < \delta_1,$$

$$|g(t) - g(b^-)| < \frac{\epsilon}{8C_m} \text{ whenever } 0 < b - t < \delta_1,$$

where $C_m = \max\{|C_1|, |C_2|, |C_3|, |C_4|\}$.

Choose

$$\delta(\xi) = \begin{cases} \delta_1 & , \text{ if } \xi = a; \\ \min\{\xi - a, c - \xi\} & , \text{ if } a < \xi < c; \\ \delta_1 & , \text{ if } \xi = c; \\ \min\{\xi - c, b - \xi\} & , \text{ if } c < \xi < b; \\ \delta_1 & , \text{ if } \xi = b. \end{cases}$$

From the choice of δ, a, b and c are associated points of any δ-fine division of $[a, b]$. Let $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ be δ-fine division of $[a, b]$. Hence,

$$\begin{aligned} &\Big|\sum_{i=1}^{n} s(\xi_i)(g(v_i) - g(u_i)) - C_1(g(a^+) - g(a)) - C_2(g(c^-) - g(a^+)) - C_3(g(c^+) - g(c^-)) \\ &\quad - C_4(g(b^-) - g(c^+)) - C_5(g(b) - g(b^-))\Big| \\ &= \Big|C_1(g(v_1) - g(a^+)) + C_2(g(v_1) - g(a^+)) + C_2(g(c^-) - g(u_j)) + C_3(g(c^-) - g(u_j)) \\ &\quad + C_3(g(v_j) - g(c^+)) + C_4(g(v_j) - g(c^+)) + C_4(g(u_n) - g(b^-)) + C_5(g(u_n) - g(b^-))\Big| \\ &\leq 2C_m\Big|(g(v_1) - g(a^+)) + (g(c^-) - g(u_j)) + (g(v_j) - g(c^+)) + (g(u_n) - g(b^-))\Big| \\ &< 2C_m 4(\frac{\epsilon}{8C_m}) = \epsilon. \end{aligned}$$

□

We remark that in the above proof, δ is a function, it is impossible to choose a constant δ.

Lemma 4.4.5. *[7] If $f \in BV_p[a, b]$, $p > 0$, then $f \in RF[a, b]$.*

Proof. Suppose that $f \notin RF[a, b]$, without lost of generality, there exists $t_0 \in [a, b]$ such that right limit of f at point t_0 does not exist. Then there exists $\epsilon > 0$, such that for every $\delta_i > 0$, there exists $[u_i, v_i] \in (t_0, t_0 + \delta_i)$ such that

$$|f(v_i) - f(u_i)| > \epsilon,$$

for every integer i. We may assume that $[u_i, v_i], i = 1, 2, ...,$ are pairwise disjoint.

Thus, we have

$$V_p(f; [a,b]) \geq \left[\sum_{i=1}^{n} |f(v_i) - f(u_i)|^p \right]^{1/p} > n^{1/p}\epsilon$$

for every integer n. Hence, $f \notin BV_p[a,b]$, it leads to a contradiction. Therefore, $f \in RF[a,b]$. □

Lemma 4.4.6. *[13, 24] If $p \geq 1$, $a = t_1 < t_2 < ... < t_n < t_{n+1} = b$ and for every $i = 2, 3, ..., n$, $f(t_i) = 0$, then*

$$V_p^p(f; [a,b]) \leq 2^p \sum_{i=1}^{n} V_p^p(f; [t_i, t_{i+1}]).$$

Proof. Let $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$. Then $f = f^+ - f^-$.

For any $x, y \in [a,b]$, note that $f^+(x) \geq 0$ and $f^+(y) \geq 0$. Hence

$$\begin{aligned} |f^+(y) - f^+(x)|^p &\leq |f^+(y)|^p \text{ or } |f^+(x)|^p \\ &\leq |f^+(y)|^p + |f^+(x)|^p \\ &= |f^+(y) - f^+(\xi)|^p + |f^+(\xi) - f^+(x)|^p, \end{aligned}$$

where $f(\xi) = 0$.

Thus, if $\{[t_i, t_{i+1}]\}_{i=1}^n$ is a partition of $[a,b]$ with $f(t_i) = 0$ for each i, we have

$$\left(V_p^p(f^+) + V_p^p(f^-)\right) \leq \sum_{i=1}^{n} \left(V_p^p(f^+; [t_i, t_{i+1}]) + V_p^p(f^-; [t_i, t_{i+1}])\right).$$

Hence,

$$\begin{aligned}
V_p^p(f) &= V_p^p(f^+ - f^-) \\
&\le \big(V_p(f^+) + V_p(f^-)\big)^p \\
&= \big(1 \cdot V_p(f^+) + 1 \cdot V_p(f^-)\big)^p \\
&\le \left[\big(1^q + 1^q\big)^{\frac{1}{q}} \big(V_p^p(f^+) + V_p^p(f^-)\big)^{\frac{1}{p}}\right]^p \\
&= 2^{p-1}\big(V_p^p(f^+) + V_p^p(f^-)\big) \\
&\le 2^{p-1} \sum_{i=1}^{n} \big(V_p^p(f^+;[t_i,t_{i+1}]) + V_p^p(f^-;[t_i,t_{i+1}])\big) \\
&\le 2^{p-1} \sum_{i=1}^{n} \big(V_p^p(f;[t_i,t_{i+1}]) + V_p^p(f;[t_i,t_{i+1}])\big) \\
&\le 2^{p} \sum_{i=1}^{n} V_p^p(f;[t_i,t_{i+1}]).
\end{aligned}$$

□

Lemma 4.4.7. *[13, p.7] Let $f \in BV_p[a,b]$, $p \ge 1$, then, given $\epsilon > 0$ and $p_1 > p \ge 1$, there is a step functions such that*

$$V_{p_1}(f - s) \le \epsilon.$$

Proof. Let $\epsilon > 0$ be given. Since $p, q \ge 1$ and $\frac{1}{p} + \frac{1}{q} > 1$, then there exist $p_1 > p \ge 1$ such that $\frac{1}{p_1} + \frac{1}{q} > 1$. Define $\lambda > 0$ by $2^{p_1} 3 \lambda^{p_1 - p} [V_p(f)]^p = \epsilon^{p_1}$. Since $f \in BV_p[a,b]$, we have $f \in RF[a,b]$ by Lemma 4.4.5. By Lemma 4.4.2, there exists a partition $a = x_1 < x_2 < ... < x_{n+1} = b$ such that for each i, for any $x, y \in (x_i, x_{i+1})$,

$$|f(y) - f(x)| \le \frac{\lambda}{2}.$$

Choose η_i such that $x_i < \eta_i < x_{i+1}$, for any $i = 1, 2, ..., n$, and define s by

$$s(\xi) = \begin{cases} f(\eta_i) & , \text{if } x_i < \xi < x_{i+1} \ , \ i = 1, 2, ..., n; \\ f(x_i) & , \text{if } \xi = x_i \ , \ i = 1, 2, ..., n+1. \end{cases}$$

Let $\{[y_k, y_{k+1}]\}_{k=1}^{m_i+1}$ be any partition of $[x_i, x_{i+1}]$. Then, we can see that

$$\begin{aligned}
&\sum_{k=1}^{m_i}|(f(y_{k+1}) - s(y_{k+1})) - (f(y_k) - s(y_k))|^p \\
&\quad =|(f(y_2) - s(y_2)) - (f(y_1) - s(y_1))|^p \\
&\qquad + \sum_{k=2}^{m_i-1} |(f(y_{k+1}) - s(y_{k+1})) - (f(y_k) - s(y_k))|^p \\
&\qquad + |(f(y_{m_i+1}) - s(y_{m_i+1})) - (f(y_{m_i}) - s(y_{m_i}))|^p \\
&\quad =|f(y_2) - s(y_2)|^p + \sum_{k=2}^{m_i-1} |f(y_{k+1}) - f(y_k)|^p + |f(y_{m_i+1}) - s(y_{m_i+1})|^p \\
&\quad \leq 3[V_p(f; [x_i, x_{i+1}])]^p.
\end{aligned}$$

Since $\{y_k\}$ is arbitrarily, so

$$V_p(f - s; [x_i, x_{i+1}]) \leq 3^{1/p} V_p(f; [x_i, x_{i+1}]).$$

Then, by Lemma 4.4.6,

$$\begin{aligned}
[V_{p_1}(f - s)]^{p_1} &\leq 2^{p_1} \sum_{i=1}^{n} [V_{p_1}(f - s; [x_i, x_{i+1}])]^{p_1} \\
&\leq 2^{p_1} \sum_{i=1}^{n} \lambda^{p_1 - p} [V_p(f - s; [x_i, x_{i+1}])]^p \\
&\leq 2^{p_1} \lambda^{p_1 - p} 3 \sum_{i=1}^{n} [V_p(f; [x_i, x_{i+1}])]^p \\
&\leq 2^{p_1} (\lambda)^{p_1 - p} 3 [V_p(f)]^p \\
&\leq \epsilon^{p_1}
\end{aligned}$$

Hence $V_{p_1}(f - s) \leq \epsilon$.

□

Now, we shall prove our main result in this chapter.

Theorem 4.4.8. *Let $f \in BV_p[a, b]$ and $g \in BV_q[a, b]$, with $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} > 1$, then f is HS-integrable with respect to g.*

Proof. Let $f \in BV_p[a,b]$ and $g \in BV_q[a,b]$, with $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} > 1$.

Let $\epsilon > 0$ be given. From Lemma 4.4.7, there is a step function s and $p_1 > p$ with $\frac{1}{p_1} + \frac{1}{q} > 1$, such that

$$V_{p_1}(f - s) \leq \epsilon.$$

By Theorem 4.4.4 and Lemma 4.4.5, $(HS) \int_a^b s\, dg$ exists. Then, there exists a positive function δ on $[a,b]$ such that for any two δ-fine divisions $D_1 = \{(\xi_l, [t_l, t_{l+1}])\}$ and $D_2 = \{(\xi_{l'}, [t_{l'}, t_{l'+1}])\}$ of $[a,b]$ we have

$$|(D_1) \sum_l s(\xi_l)(g(t_{l+1}) - g(t_l)) - (D_2) \sum_{l'} s(\xi_{l'})(g(t_{l'+1}) - g(t_{l'}))| < \epsilon.$$

Hence, from Corollary 4.3.11 and Lemma 4.4.7, we can see that,

$$\begin{aligned}
&|(D_1) \sum_l f(\xi_l)(g(t_{l+1}) - g(t_l)) - (D_2) \sum_{l'} f(\xi_{l'})(g(t_{l'+1}) - g(t_{l'}))| \\
=&|(D_1) \sum_l (f(\xi_l) - s(\xi_l))(g(t_{l+1}) - g(t_l)) \\
&+ (D_1) \sum_l s(\xi_l)(g(t_{l+1}) - g(t_l)) - (D_2) \sum_{l'} s(\xi_{l'})(g(t_{l'+1}) - g(t_{l'})) \\
&- (D_2) \sum_{l'} (f(\xi_{l'}) - s(\xi_{l'}))(g(t_{l'+1}) - g(t_{l'}))| \\
=&|(D_1) \sum_l (f(\xi_l) - s(\xi_l))(g(t_{l+1}) - g(t_l)) - (D_2) \sum_{l'} (f(\xi_{l'}) - s(\xi_{l'}))(g(t_{l'+1}) - g(t_{l'}))| \\
&+ |(D_1) \sum_l s(\xi_l)(g(t_{l+1}) - g(t_l)) - (D_2) \sum_{l'} s(\xi_{l'})(g(t_{l'+1}) - g(t_{l'}))| \\
\leq&2\{1 + \zeta(\frac{1}{p_1} + \frac{1}{q})\} V_{p_1}(f - s) V_q(g) + \epsilon \\
\leq&2\{1 + \zeta(\frac{1}{p_1} + \frac{1}{q})\} \epsilon V_q(g) + \epsilon.
\end{aligned}$$

Hence $(HS) \int_a^b f\, dg$ exists. □

From now onwards, if $f \in BV_p[a,b], g \in BV_q[a,b]$, where $\frac{1}{p} + \frac{1}{p} > 1$, $p, q \geq 1$, then the Henstock-Stieltjes integral $(HS) \int_a^b f\, dg$ is called the Henstock-Young integral, and denoted by $(HY) \int_a^b f\, dg$.

4.5 Convergence theorems

In this section, we shall prove some convergence theorems.

Definition 4.5.1 (Two-norm convergence)**.** A sequence $\{f_n\}$ of functions defined on $[a,b]$ is said to be two-norm convergent to f in $BV_p[a,b]$ if $f_n \in BV_p[a,b]$, for all $n = 1, 2, ...$, and

(i) f_n is uniformly convergent to f on $[a,b]$.

(ii) $V_p(f_n) \leq A$ for every $n = 1, 2, ...$

In symbols, we denote the two-norm convergence by $f_n \twoheadrightarrow f$.

It is clear that $BV_p[a,b]$ is complete under two-norm convergence, i.e., if $f_n \in BV_p[a,b]$, $n = 1, 2, ...$, and $f_n \twoheadrightarrow f$, then $f \in BV_p[a,b]$.

Theorem 4.5.2. *If a sequence $\{f^{(n)}\}$ is two-norm convergent to f in $BV_p[a,b]$ and $g \in BV_q[a,b]$, with $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} > 1$, then $(HY)\displaystyle\int_a^b f\ dg$ exists and*

$$\lim_{n\to\infty} (HY)\int_a^b f^{(n)}\ dg = (HY)\int_a^b f\ dg.$$

Proof. In this proof, $(HY)\displaystyle\int_a^b f\ dg$ is denoted by $\displaystyle\int_a^b f\ dg$. Let $\epsilon > 0$ be given. Let $\{f^{(n)}\}$ be two-norm convergent to f in $BV_p[a,b]$ and $g \in BV_q[a,b]$, with $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} > 1$. First, $\displaystyle\int_a^b (f^{(n)} - f)\ dg$ exists. Thus, there exists a positive function δ_n on $[a,b]$ such that for every δ_n-fine division $D = \{([t_i, t_{i+1}], \xi_i)\}$ of $[a,b]$,

$$\left| (\int_a^b (f^{(n)} - f)\ dg) - (D)\sum (f^{(n)}(\xi_i) - f(\xi_i))(g(t_{i+1}) - g(t_i)) \right| \leq \epsilon. \qquad (4.5.3)$$

Let

$$V_p(f^{(n)} - f) \leq A \text{ for every } n \text{ and } V_q(g) = B.$$

Since $f^{(n)} \twoheadrightarrow f$, there is a positive integer N such that for every $n \geq N$, we have

$$\sup_{t\in[a,b]} \{|f^{(n)}(t) - f(t)|\} = ||f^{(n)} - f||_\infty \leq \frac{\epsilon}{2}. \tag{4.5.4}$$

Choose a fixed $p_1 > p$ such that $\frac{1}{p_1} + \frac{1}{q} > 1$, then $f^{(n)} - f \in BV_{p_1}[a,b]$. Furthermore, for $n \geq N$ and a δ_n-fine division $D = \{([t_i, t_{i+1}], \xi_i)\}$ of $[a,b]$, by Theorem 4.3.9, inequalities (4.5.3) and (4.5.4).

$$\begin{aligned}
&\Big|\int_a^b f^{(n)}\ dg - \int_a^b f\ dg\Big| \\
\leq &\Big|(f^{(n)}(a) - f(a))(g(b) - g(a))\Big| + \Big|\int_a^b (f^{(n)} - f)\ dg - (f^{(n)}(a) - f(a))(g(b) - g(a))\Big| \\
\leq &\frac{\epsilon}{2}B + \Big|\int_a^b (f^{(n)} - f)\ dg - (D)\sum(f^{(n)}(\xi_i) - f(\xi_i))(g(t_{i+1}) - g(t_i))\Big| \\
&+ \Big|(D)\sum(f^{(n)}(\xi_i) - f(\xi_i))(g(t_{i+1}) - g(t_i)) - \sum(f^{(n)}(t_{i+1}) - f(t_{i+1}))(g(t_{i+1}) - g(t_i))\Big| \\
&+ \Big|\sum(f^{(n)}(t_{i+1}) - f(t_{i+1}))(g(t_{i+1}) - g(t_i)) - (f^{(n)}(a) - f(a))(g(b) - g(a))\Big| \\
\leq &\frac{\epsilon}{2}B + \epsilon + 2\Big\{1 + \zeta(\frac{1}{p_1} + \frac{1}{q})\Big\}V_{p_1}(f^{(n)} - f)V_q(g) + \Big\{1 + \zeta(\frac{1}{p_1} + \frac{1}{q})\Big\}V_{p_1}(f^{(n)} - f)V_q(g) \\
\leq &\frac{\epsilon}{2}B + \epsilon + 3\Big\{1 + \zeta(\frac{1}{p_1} + \frac{1}{q})\Big\}\epsilon^{(p_1 - p/p_1)}V_p^{p/p_1}(f^{(n)} - f)V_q(g) \\
= &\frac{\epsilon}{2}B + \epsilon + 3\Big\{1 + \zeta(\frac{1}{p_1} + \frac{1}{q})\Big\}\epsilon^{(p_1 - p/p_1)}A^{p/p_1}B.
\end{aligned}$$

Hence, $\lim_{n\to\infty} (HY)\int_a^b f^{(n)}\ dg = (HY)\int_a^b f\ dg.$ □

Using the idea of the above proof, we have

Theorem 4.5.5. *If $f \in BV_p[a,b]$ and $\{g^{(n)}\}$ is two-norm convergent to g in $BV_q[a,b]$, with $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} > 1$, then $(HY)\int_a^b f\ dg$ exists and*

$$\lim_{n\to\infty} (HY)\int_a^b f\ dg^{(n)} = (HY)\int_a^b f\ dg.$$

Proof. In this proof, $(HS)\displaystyle\int_a^b f\ dg$ is denoted by $\displaystyle\int_a^b f\ dg$. Let $\epsilon > 0$ be given. Let $\{g^{(n)}\}$ be two-norm convergent to g in $BV_q[a,b]$ and $f \in BV_p[a,b]$, with $p, q \geq 1$ and $\frac{1}{p}+\frac{1}{q} > 1$. First, $\displaystyle\int_a^b f\ d(g^{(n)}-g)$ exists. Thus, there exists a positive function δ_n on $[a,b]$ such that for every δ_n-fine division $D = \{([t_i, t_{i+1}], \xi_i)\}_{i=1}^n$ of $[a,b]$,

$$\left|(\int_a^b f\ d(g^{(n)}-g)) - (D)\sum f(\xi_i)((g^{(n)}(t_{i+1}) - g(t_{i+1})) - (g^{(n)}(t_i) - g(t_i)))\right| \leq \epsilon.$$

Let

$$|f(a)| + V_p(f) = A \text{ and } V_q(g^{(n)} - g) \leq B \text{ for every } n$$

Since $g^{(n)} \twoheadrightarrow g$, there is a positive integer $N \in \mathbb{N}$ such that for every $n \geq N$, $\sup_{t\in[a,b]} \{|g^{(n)}(t) - g(t)|\} = \|g^{(n)} - g\|_\infty \leq \dfrac{\epsilon}{2}$. Hence, for $q_1 > q$ such that $\frac{1}{p}+\frac{1}{q_1} > 1$, we have $g^{(n)} - g \in BV_{p_1}[a,b]$ and

$$\begin{aligned}
&\left|\int_a^b f\ dg^{(n)} - \int_a^b f\ dg\right| \\
\leq&\left|f(a)((g^{(n)}(b) - g(b)) - (g^{(n)}(a) - g(a)))\right| \\
&+\left|\int_a^b f\ d(g^{(n)}-g) - f(a)((g^{(n)}(b) - g(b)) + (g^{(n)}(a) - g(a)))\right| \\
\leq&A\epsilon + \left|\int_a^b f\ d(g^{(n)}-g) - (D)\sum f(\xi_i)((g^{(n)}(t_{i+1}) - g(t_{i+1})) - (g^{(n)}(t_i) - g(t_i)))\right| \\
&+\Bigg|(D)\sum f(\xi_i)((g^{(n)}(t_{i+1}) - g(t_{i+1})) - (g^{(n)}(t_i) - g(t_i))) \\
&-\sum f(t_{i+1})((g^{(n)}(t_{i+1}) - g(t_{i+1})) - (g^{(n)}(t_i) - g(t_i)))\Bigg| \\
&+\Bigg|\sum f(t_{i+1})((g^{(n)}(t_{i+1}) - g(t_{i+1})) - (g^{(n)}(t_i) - g(t_i))) \\
&-f(a)((g^{(n)}(b) - g(b)) - (g^{(n)}(a) - g(a)))\Bigg| \\
\leq&A\epsilon + \epsilon + 2\left\{1 + \zeta(\frac{1}{p}+\frac{1}{q_1})\right\}V_p(f)V_{q_1}(g^{(n)} - g) + \left\{1 + \zeta(\frac{1}{p}+\frac{1}{q_1})\right\}V_p(f)V_{q_1}(g^{(n)} - g) \\
\leq&A\epsilon + \epsilon + 3\left\{1 + \zeta(\frac{1}{p}+\frac{1}{q_1})\right\}\epsilon^{(q_1 - q/q_1)}V_p(f)V_q^{q/q_1}(g^{(n)} - g) \\
=&A\epsilon + \epsilon + 3\left\{1 + \zeta(\frac{1}{p}+\frac{1}{q_1})\right\}\epsilon^{(q_1 - q/q_1)}AB^{q/q_1}.
\end{aligned}$$

Hence, $\lim_{n\to\infty} (HY) \int_a^b f\ dg^{(n)} = (HY) \int_a^b f\ dg.$ $\square$

Again following the proof of Theorem 4.5.2, We have

$$\lim_{n\to\infty} \Big((HY) \int_a^b f^{(n)}\ dg^{(n)} - (HY) \int_a^b f\ dg^{(n)} \Big) = 0.$$

Hence, we have the following Theorem.

Theorem 4.5.6. *If $\{f^{(n)}\}$ and $\{g^{(n)}\}$ are two-norm convergent to f and g in $BV_p[a,b]$ and $BV_q[a,b]$, respectively, with $p, q \geq 1$ and $\frac{1}{p}+\frac{1}{q} > 1$, then $(HY) \int_a^b f\ dg$ exists and*

$$\lim_{n\to\infty} (HY) \int_a^b f^{(n)}\ dg^{(n)} = (HY) \int_a^b f\ dg.$$

Similar convergence theorems have been proved by K.K.Aye in [1, p.71].

4.6 Examples

In this section, we shall consider Fractional Brownian Motions.

Definition 4.6.1. A fractional Brownian motion $B_t^H : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}$ with Hurst parameter $H \in (0,1]$ is a process with the following properties:

(a) for any $t_1, t_2, ..., t_n \in \mathbb{R}_0^+$, $(B_{t_1}^H, B_{t_2}^H, ..., B_{t_n}^H)$ is an n-dimensional normal distribution with mean zero;

(b) $Cov\big(B_t^H, B_s^H\big) = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big)$, for any $t, s \in \mathbb{R}_0^+$, where $Cov\big(B_t^H, B_s^H\big)$ is the covariance of B_t^H and B_s^H, i.e., $Cov\big(B_t^H, B_s^H\big) = \mathbf{E}\big(B_t^H B_s^H\big)$ (note that $\mathbf{E}(B_t^H) = 0$ for any t);

(c) B^H has continuous paths, i.e., for almost all $\omega \in \Omega$, $B_t^H(\omega)$ is continuous in t .

It is known that

(i) when $H = \frac{1}{2}$, B_t^H is a Brownian motion mentioned in Section 1.3 ;

(ii) $B_t^H(\omega) = \frac{1}{\Gamma(H+\frac{1}{2})}\left\{ \int_{-\infty}^0 \{(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\}\ dB_t(\omega) + \int_0^t (t-s)^{H-\frac{1}{2}}\ dB_s(\omega) \right\}$;

(iii) for almost all $\omega \in \Omega$, $B_t^H(\omega) \in BV_p[a,b]$, for any p with $\frac{1}{H} < p$,

where $\Gamma(H + \frac{1}{2})$ is a Gamma function at $H + \frac{1}{2}$.

From (ii), fractional Brownian motion B_t^H is a process of p-bounded variation on $[a,b]$, where $p > \frac{1}{H}$, i.e., for almost all $\omega \in \Omega$, $B_t^H(\omega) \in BV_p[a,b]$. We can define $(HY)\int_a^b f_t(\omega)\ dB_t^H(\omega)$ for almost all $\omega \in \Omega$. Hence, $f_t(\omega)$ is integrable with respect to $B_t^H(\omega)$ for almost all $\omega \in \Omega$, if for each $\omega \in \Omega$, $f_t(\omega) \in BV_q[a,b]$ and $\frac{1}{p} + \frac{1}{q} > 1$, with $q > 0$. Suppose $\frac{1}{H} < p < \frac{1}{H-\eta}$ where $0 < \eta < H$. Then $0 < H - \eta < \frac{1}{p} < H$. Now we shall illustrate how to choose q. Let $\frac{1}{q} = 1 - (H - \eta) = 1 - H + \eta$. Then $\frac{1}{p} + \frac{1}{q} > H - \eta + 1 - (H - \eta) = 1$ and $q > 1$. We also can choose q such that $1 > \frac{1}{q} > 1 - (H - \eta)$, then $\frac{1}{p} + \frac{1}{q} > 1$ and $q > 1$.

Now, consider $H = \frac{1}{2}$, i.e., $B_t^H = B_t$ is a Brownian motion. Then choose p such that $\frac{1}{H} = 2 < p < \frac{1}{H-\eta}$, where η is a small positive number and choose q such that $q = \frac{1}{1-H+\eta} = \frac{1}{\frac{1}{2}+\eta} = \frac{2}{1+2\eta}$, i.e., q is slightly less than 2. Hence for a Brownian motion B_t, $(HY)\int_a^b f_t(\omega)\ dB_t(\omega)$ exists if $f(\omega) \in BV_q[a,b]$ for almost all ω, where q is slightly less than 2.

Recall that the Itô -integral of f is defined using L^2-norm. It is not defined with respect to a Brownian path $B_t(\omega)$, where ω is fixed. It is known that if $\mathbf{E}(f_t^2)$ is Lebesgue integrable on $[a,b]$, then f is Itô -integrable on $[a,b]$. On the other hand if we fix ω, and consider $(HY)\int_a^b f_t(\omega)\ dB_t(\omega)$, then we need a stronger condition on $f_t(\omega)$, the condition is that $f_t(\omega) \in BV_q[a,b]$, where q is slightly less than 2.

We remark that when we consider fractional Brownian motion B_t^H, when $H \neq \frac{1}{2}$ the HY integral is useful since we do not have the corresponding Itô -integral for B_t^H, when $H \neq \frac{1}{2}$. Recall that for the Itô -integral, the orthogonal increment property $\mathbf{E}\big((B_v - B_u)(B_t - B_s)\big) = 0$, where $u < v \leq s < t$, plays an important role. However, this property does not hold for B_t^H, when $H \neq \frac{1}{2}$. Therefore, we do not have the corresponding Itô integral for B_t^H, when $H \neq \frac{1}{2}$.

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